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- Semi-parametric estimation
- Non-parametric estimation
- Random fields and regularity issues

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- Definition
- Discrete wavelet transform (DWT)
- Asymptotic behavior of the DWT
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- The Infinite Source Poisson process
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Modélisation et estimation de la dépendance et de la régularité

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June 19, 2007

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Color code

We will adopt the following color convention:

- X Variable of interest (observed)
- θ Parameter of interest (unknown)
- Z Hidden variable (indirectly observed)
- β Tunable parameter (chosen)

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Semi-parametric estimation

We have considered 3 semi-parametric estimation problems:

- 1 Estimation of the frequency of an irregularly sampled periodic function, [8].
- 2 Wavelet methods for estimating the long memory parameter of Gaussian and linear processes [13, 11, 12];
- 3 Estimation of the tail index of the sessions durations in an $M/G/\infty$ type process [6].

Frequency estimation

In [8] the model

$$Y_j = s(X_j) + \epsilon_j, \quad j = 1, \dots, n,$$

is considered, where

- (i) s is an unknown periodic function,
- (ii) X_j is a renewal process (having i.i.d. positive increments with finite means),
- (iii) $\{\epsilon_j, j \geq 1\}$ is a Gaussian white noise, independent of $\{X_j, j \geq 1\}$.

We consider the problem of estimating the frequency of s when $(X_j, Y_j), j = 1, \dots, n$ is observed. We use the Lomb-Scargle periodogram, exhibit a consistent estimator and show a central limit theorem for this estimator.

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Wavelet estimation of the memory parameter

In [13],

the second order properties of the discrete wavelet transform of a long memory process and the mean square error of a wavelet log-regression estimator of the long memory parameter of a Gaussian process are studied.

In [11],

a central limit theorem for this estimator is obtained.

In [12],

a different wavelet estimator is considered, which corresponds to a Whittle contrast approach in the wavelet domain. In this paper the rate of convergence is derived for a wide class of long memory linear processes and a central limit theorem is obtained in the Gaussian case.

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Tail estimation

In [6], we estimate the tail index of the durations of the pulses of a shot-noise

$$X_t = \sum_k \mathbf{U}_k \mathbb{1}(\mathbf{T}_k \leq t < \mathbf{T}_k + \eta_k),$$

where $\{\mathbf{T}_k\}$ are Poisson arrival times and $\{(\mathbf{U}_k, \eta_k)\}$ are i.i.d. marks, independent of the arrival times and satisfying

$$\mathbb{E}[\mathbf{U}_0^p \mathbb{1}_{\eta_0 > t}] = L_p(t) t^{-\alpha},$$

with $p = 0, 1, \dots, 4$, L_p slowly varying as $t \rightarrow \infty$ and $\alpha \in (0, 2)$. The estimator only depends on (possibly discretely sampled) observations of $\{X_t, t = 0, \dots, T\}$. It is related to [12] as it relies on the long memory behavior of this process. The consistency and the rate of covariance are investigated.

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Non-parametric estimation

We have considered 3 non-parametric estimation problems:

- 1 Estimation of the density of the pulse energy distribution of a shot noise process [15, 10].
- 2 Estimation of the density of a mixing distribution for mixtures on a discrete state space [14].
- 3 Estimation of a tvAR process [9];

Density of the pulse energy of a shot-noise process

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In [15, 10], we consider a shot-noise model with Poisson arrivals

$$S(t) = \sum_k \mathbf{Z}_k(t - \mathbf{T}_k),$$

where $\{\mathbf{Z}_k(t), t \geq 0\}$ are i.i.d. processes and independent of the Poisson arrivals $\{\mathbf{T}_k\}$. We wish to estimate the distribution of the variables

$$\mathbf{Y}_k = \int_0^\infty \mathbf{Z}_k(u) du$$

from observations of $\{S(t), t \in (0, T)\}$. We study two estimators based on the cycles of the process S (idle and busy periods).

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Density of a mixing distribution

In [14], we study a projection estimator of the density f of a mixing distribution with respect to a given dominating measure ν , obtained from n i.i.d. random variables X_1, \dots, X_n having the mixture distribution

$$\pi_f = \int \pi_\theta f(\theta) \nu(d\theta),$$

where $\{\pi_\theta\}$ is a known family of mixand distributions. We focus on the case where this family is a discrete distribution and obtain minimax bounds for estimating f .

Time-varying AR processes

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In [9], we consider a time varying auto-regressive (TVAR) model defined by

$$X_{k+1,n} = \theta_{k,n}^t \mathbf{X}_{k,n} + \sigma_{k+1,n} \epsilon_{k+1,n} ,$$

where

- (i) we denoted $\mathbf{X}_{k,n} = [X_{k,n} \ \dots \ X_{k-p,n}]^T$,
- (ii) the array $\{\epsilon_{k,n}, k \in \mathbb{N}, n \geq 1\}$ is a centered white noise with unit variance,
- (iii) the time varying coefficients are given by $\theta_{k,n} = \theta(k/n)$ with $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}^p$ and $\sigma_{k,n} = \sigma(k/n)$ with $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

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We give some conditions for the stability of this model as $n \rightarrow \infty$, provide a lower bound of the **minimax** rate and we study a recursive estimator from the observations $X_{k,n}$, $k = 1, \dots, n$ defined by

$$\hat{\theta}_{k+1,n} = \hat{\theta}_{k,n} + \mu \frac{(X_{k+1,n} - \hat{\theta}_{k,n}^T \mathbf{X}_{k,n}) \mathbf{X}_{k,n}}{1 + \mu \|\mathbf{X}_{k,n}\|^2}, \quad (1)$$

where $\mu > 0$ is the step-size tuned by the user and $\|\cdot\|$ denotes the Euclidean norm.

Under mild assumption, we show that this estimator achieves the minimax rate under certain weak smoothness assumptions and show how it can be adapted to stronger smoothness assumptions.

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Random fields and regularity issues

We have studied the following random fields.

- 1 Random wavelet series [1].
- 2 Linear fractional stable sheet [3, 2].
- 3 Scaling dead leaves model [7, 4].

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Random wavelet series

Define a **random wavelet series** as

$$X(t) = \sum_{\lambda \in \Lambda} \epsilon_{\lambda} \psi_{\lambda}(t), \quad (2)$$

where

- ① Λ is a countable set of indices, each index $\lambda \in \Lambda$ corresponding to a subset of \mathbb{R}^d ,
- ② $\{\epsilon_{\lambda}\}$ is a sequence of random variables,
- ③ $\{\psi_{\lambda}\}$ is a sequence of $\mathbb{R}^d \rightarrow \mathbb{R}$ functions, conveniently normalized and localized using λ .

In [1], we are interested in the Hausdorff dimension of the graph of such random processes. A new formulation of the Frostman criterion is proposed for bounding this dimension from below and applied when

$$\lambda = 2^{-j}[k, k+1), \quad \psi_{\lambda} = \psi(2^j t - k), \quad k \in \mathbb{Z}, j \geq 0,$$

and $\{\epsilon_{\lambda}\}$ are independent.

Linear fractional stable sheet

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The linear fractional stable sheet $((N, 1)$ -LFSS) with parameters $\alpha \in (0, 2]$ et $H = (H_1, \dots, H_N) \in (0, 1)^N$ is the real-valued α -stable field

$$X(t) = \int_{\mathbb{R}^N} \prod_{l=1}^N \left\{ (t_l - s_l)_+^{H_l - 1/\alpha} - (-s_l)_+^{H_l - 1/\alpha} \right\} dZ_\alpha(s),$$

where $\{Z_\alpha(s), s \in \mathbb{R}^N\}$ is an α -stable Lévy sheet with given skewness intensity $\beta : \mathbb{R}^N \rightarrow [-1, 1]$.

In [3], we give a random wavelet series representation of the $(N, 1)$ -LFSS and use it for studying the modulus of continuity.

We also show the existence and joint continuity of the local time of the (N, d) -LFSS in [2].

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The dead leaves model

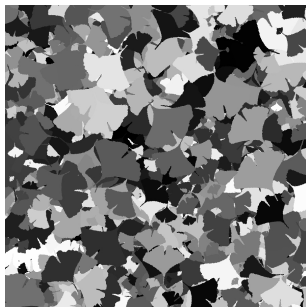


Figure: The dead leaves model

In [4], we provide some new results on the dead leaves model, a model obtain by randomly superimposing random object in the space, resulting in a random tessellation. For instance we completely characterize the distribution of its boundary.

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Scaling dead leaves model

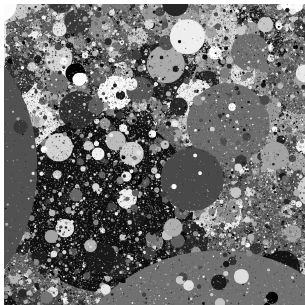


Figure: Scaling dead leave model

In [7], we investigate some **scaling** dead leaves models obtained by letting converge a dead leaves model as the objects sizes distribution approaches a power law. We then study the regularity properties of the limit in the most interesting cases.

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Definition

Let $X = \{X(t), t \in \mathbb{Z}\}$ be a process with finite variance. It has memory parameter d if, around the null frequency, X has a spectral density behaving as $|\lambda|^{-2d}$.

The spectral density must be defined in a **wide sense** to allow the **non-stationary** case $d \geq 1/2$. In this case, the spectral density is only defined for high pass linear filters of X , for instance $(\mathbf{I} - \mathbf{B})^k X$ with $k > d - 1/2$.

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A definition for $d \in \mathbb{R}$

Let V_d denote the set of real valued sequences $(h_k) \in \mathbb{R}^{\mathbb{Z}}$ with finite support such that

$$\int_{-\pi}^{\pi} |h^*(\lambda)|^2 |\lambda|^{-2d} d\lambda < \infty, \text{ where } h^*(\lambda) = \sum_k h_k e^{ik\lambda}.$$

Definition

The process $\left\{ X(h) := \sum_k h_k X_k, h \in V_d \right\}$ has memory parameter d if there exists a measure ν defined on $(-\pi, \pi]$, finite away of 0 and such that $d\nu/d\lambda \asymp |\lambda|^{-2d}$ in a neighborhood of 0, such that for all $(h_k) \in V_d$,

$$\text{Var}(X(h)) = \int_{-\pi}^{\pi} |h^*(\lambda)|^2 d\nu(\lambda). \quad (3)$$

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Limit variance of a sequence of linear combinations

Let $(s_\ell)_{\ell \geq 1}$ be a positive sequence tending to ∞ .

Let $(h_{\ell,k})_{\ell \geq 1, k \in \mathbb{Z}}$ be such that, for some $C, \alpha_1, \alpha_2 > 0$,

$$h_\ell^*(\lambda/s_\ell) \rightarrow h^*(\lambda) \quad \text{as } \ell \rightarrow \infty, \quad \lambda \in \mathbb{R},$$

$$|h_\ell^*(\lambda/s_\ell)| \leq C |\lambda|^{d+\alpha_1-1/2} (1+|\lambda|)^{-\alpha_1-\alpha_2}, \quad |\lambda| \leq \ell\pi.$$

Then, as $\ell \rightarrow \infty$,

$$\text{Var}(X(h_\ell)) \asymp s_\ell^{2d-1} \int_{-\infty}^{\infty} |h^*(\lambda)|^2 |\lambda|^{-2d} d\lambda. \quad (4)$$

Exemple 1 (sample mean)

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Set $h_{\ell,k} = \ell^{-1} \mathbb{1}(1 \leq k \leq \ell)$. Then

$$h_{\ell}^*(\lambda/\ell) \rightarrow 2e^{i\lambda/2} \sin(\lambda/2)/\lambda, \quad \text{as } \ell \rightarrow \infty, \quad \lambda \in \mathbb{R},$$

$$|h_{\ell}^*(\lambda/\ell)| \leq C(1 + |\lambda|)^{-1}, \quad |\lambda| \leq \ell\pi.$$

Using (4), if $-1/2 < d < 1/2$,

$$\text{Var} \left(\ell^{-1} \sum_{k=1}^{\ell} X_k \right) \asymp \ell^{2d-1}.$$

Exemple 1 in a figure

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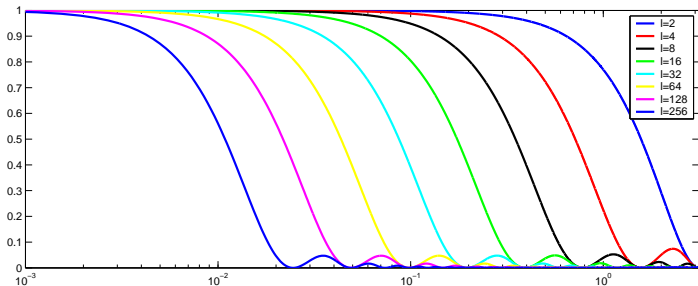
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We plot $|h_\ell^*(\lambda)|^2$.



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Discrete wavelet transform (DWT)

Let $\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t - k)$ where ψ is a wavelet. For $x(t)$ defined on continuous time $t \in \mathbb{R}$, the DWT is defined by

$$W_{j,k} = \int_{-\infty}^{\infty} x(t)\psi_{j,k}(t) dt, \quad j \in \mathbb{Z}, k \in \mathbb{Z}.$$

For x_k defined on discrete time $k \in \mathbb{Z}$, we use the interpolated version

$$x(t) = \sum_{l \in \mathbb{Z}} x_l \phi(t - l), \quad t \in \mathbb{R},$$

resulting in $W_{j,k} = \sum_{l \in \mathbb{Z}} h_{j,2^j k - l} x_l = [\downarrow^j (h_{j,\cdot} * x)]_k$.

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Exemple 2 (DWT)

If ψ has M vanishing moments and ϕ interpolates polynomials of degree $M - 1$ and $|\hat{\psi}(\lambda)| \leq C (1 + |\lambda|)^{-\alpha}$, then

$$2^{-j/2} \mathbf{h}_j^*(2^{-j}\lambda) \rightarrow \hat{\phi}(0) \overline{\hat{\psi}(\lambda)}, \quad \text{as } j \rightarrow \infty, \quad \lambda \in \mathbb{R},$$

$$2^{-j/2} |\mathbf{h}_j^*(2^{-j}\lambda)| \leq C |\lambda|^M (1 + |\lambda|)^{-\alpha}, \quad |\lambda| \leq \ell\pi.$$

Using (4), if $1/2 - \alpha < d < M + 1/2$,

$$\text{Var}(W_{j,k}) \asymp 2^{2dj}.$$

Exemple 2 in a figure

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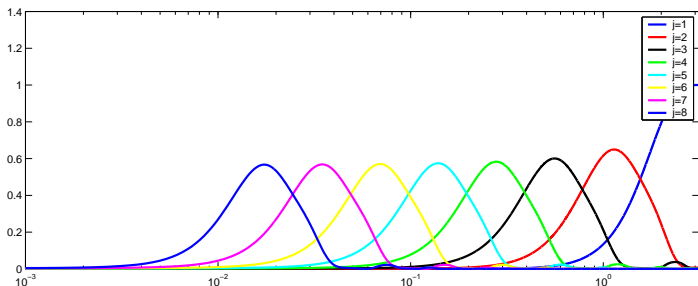
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We plot $2^{-j} \left| h_j^*(\lambda) \right|^2$ when ϕ and ψ are the Daubechies wavelets with $M = 2$.



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Asymptotic behavior

For all $j \geq u \geq 0$,

$$\mathbf{W}_{j,k}(u) := \begin{bmatrix} W_{j,k} \\ W_{j-u,k2^u} \\ W_{j-u,k2^u+1} \\ \vdots \\ W_{j-u,(k+1)2^u-1} \end{bmatrix}, \quad k \in \mathbb{Z},$$

is stationary if $M > d - 1/2$ and **weakly dependent**¹ if $M \geq d$.

Moreover, as $j \rightarrow \infty$, the spectral density of

$$\text{Var}(W_{j,0})^{-1/2} \mathbf{W}_{j,k}(u), \quad k \in \mathbb{Z}, \quad (5)$$

converges to the one of a **weakly dependent** process, thanks to the **decimation** effect.

¹here means that it has a bounded spectral density 

DWT of a self-similar process

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The DWT of a pure $(d - 1/2)$ -self similar process (defined in continuous time) satisfies

$$\text{Cov} \left(W_{j,k}^{(d)}, W_{j',k'}^{(d)} \right) = \int_{-\infty}^{\infty} \hat{\psi}_{j,k}(\lambda) \overline{\hat{\psi}_{j',k'}(\lambda)} |\lambda|^{-2d} d\lambda .$$

The limit spectral density of (5) is the one of

$$\text{Var} \left(W_{0,0}^d \right)^{-1/2} \mathbf{W}_{0,k}^{(d)}(u), \quad k \in \mathbb{Z} .$$

Unfortunately, if $\{\psi_{j,k}\}$ is an orthonormal system, these coefficients are **not** uncorrelated, unless $d = 0$.

Wavelet estimators of d

If X has a memory parameter d , and ϕ and ψ are chosen so that $1/2 - \alpha < d < M + 1/2$, we have seen that

$\text{Var}(W_{j,k}) \asymp 2^{2dj}$ as $j \rightarrow \infty$.

Under standard semi-parametric type of assumptions, the \asymp can be made more precise, e.g.

$$\left| \text{Var}(W_{j,k}) - \sigma^2 2^{2dj} \right| \leq C 2^{-\beta j} 2^{2dj}. \quad (6)$$

Simple wavelet estimators are based on the empirical variances

$$\hat{\sigma}_j^2 = n_j^{-1} \sum_{k=0}^{n_j-1} W_{j,k}^2,$$

where n_j is the number of available wavelet coeff. at scale j from n observations ($n_j \asymp n2^{-j}$).

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Asymptotic results

The basic tool is to study the joint convergence of (conveniently normalized)

$$\hat{\sigma}_j^2 - \sigma^2 2^{2dj} = [\hat{\sigma}_j^2 - \text{Var}(W_{j,k})] \text{ (fluctuation term)} \\ + [\text{Var}(W_{j,k}) - \sigma^2 2^{2dj}] \text{ (bias term),}$$

for $J_0 \leq j \leq J_{\max}$ as J_0 and $n \rightarrow \infty$.

The **Bias term** is given by (6);

The weak dependence structure allows to study the asymptotic behavior of the **fluctuation term** if X is a **linear process**

$$X_k = \sum_{t \in \mathbb{Z}} a_{k-t} \xi_t,$$

with $\{\xi_t\}$ i.i.d. and $\mathbb{E}[\xi_0^4] < \infty$.

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In this case one finds, as j and $n_j \rightarrow \infty$,

$$2^{-2dj} \hat{\sigma}_j^2 - \sigma^2 = O_P \left(n_j^{-1/2} \right) + O \left(2^{-\beta j} \right) .$$

Define the Whittle estimator \hat{d} as the maximizer of

$$d \mapsto \log \left(\sum_{j=J_0}^{J_1} w_j 2^{-2dj} \hat{\sigma}_j^2 \right) + 2 \log(2) d \sum_{j=J_0}^{J_1} j w_j ,$$

where $w_j \propto n_j$ with $\sum_{j=J_0}^{J_1} w_j = 1$.

One finds similar asymptotic results as for semi-parametric Fourier estimators except that

the asymptotic variance depend on d .

<http://www.tsi.enst.fr/~roueff/hdr>

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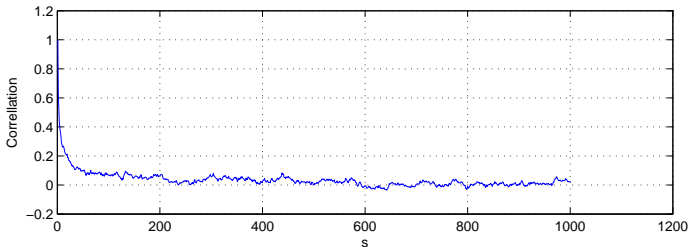
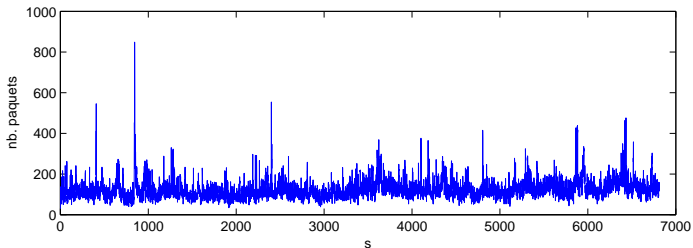
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Teletraffic data

Around 2 hours IP traffic record aggregated every second.



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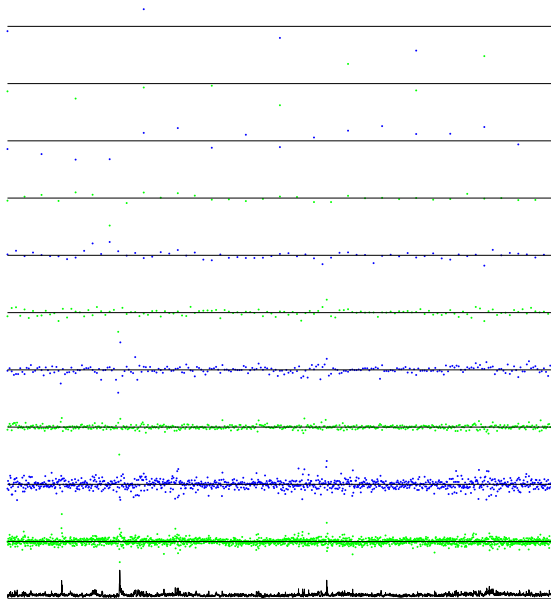
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And its DWT



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The Infinite source Poisson model

Consider the shot-noise

$$X_t = \sum_k \mathbf{U}_k \mathbb{1}(\mathbf{T}_k \leq t < \mathbf{T}_k + \eta_k),$$

where $\{\mathbf{T}_k\}$ are Poisson arrival times and $\{(\mathbf{U}_k, \eta_k)\}$ are i.i.d. marks, independent of the arrival times and satisfying

$$\mathbb{E}[\mathbf{U}_0^p \mathbb{1}_{\eta_0 > t}] = L_p(t) t^{-\alpha},$$

with $p = 0, 1, \dots, 4$, L_p slowly varying as $t \rightarrow \infty$ and $\alpha \in (0, 2)$.

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Three possible observation schemes

- ① We observe the **continuous** path X_t for all $t \in [0, T]$.
- ② We observe the **discrete** sample path X_t for all $t = 1, 2, \dots, T$.
- ③ We observe **discrete** local averages

$$Y_k = \int_k^{k+1} X_t dt$$

for all $k = 0, 1, \dots, T$.

In fact the two last cases can be treated similarly. Let us consider the discrete time case: we wish to estimate α from the observation X_t , $t = 1, 2, \dots, T$. This is an **inverse problem** in the sense that we do not observe $(\mathbf{U}_k, \eta_k)_k$ **directly**.

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Stationarity issues

If $\mathbb{E}[\eta_0] < \infty$ ($\Leftrightarrow \alpha > 1$) then a stationary version of X can be defined by taking a uniform intensity on \mathbb{R} for the arrivals $\{\mathbf{T}_k\}$.

If $\mathbb{E}[\eta_0] = \infty$ ($\Leftrightarrow \alpha < 1$), this version is only defined for

$$X(\mathbf{h}.) = \sum_k \mathbf{U}_k \sum_{\mathbf{T}_k \leq t < \mathbf{T}_k + \eta_k} \mathbf{h}_t \left(= \sum_t \mathbf{h}_t X_t \right),$$

is defined for (\mathbf{h}_k) with finite support and $\sum_t \mathbf{h}_t = 0$, in which case

$$\text{Var}(X(\mathbf{h}.)) = \mathbb{E} \left[\mathbf{U}_0^2 \sum_{t,t'} \mathbf{h}_t \mathbf{h}_{t'} (\eta_0 - |t - t'|)_+ \right].$$

Observe that, for η_0 large enough,

$$\sum_{t,t'} \mathbf{h}_t \mathbf{h}_{t'} (\eta_0 - |t - t'|)_+ = - \sum_{t,t'} |t - t'| \mathbf{h}_t \mathbf{h}_{t'}.$$

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Memory parameter

One finds

$$\text{Var}(X(\mathbf{h}_\cdot)) = \int_{-\infty}^{\infty} |h^*(\lambda)|^2 \frac{\mathbb{E}[\mathbf{U}_0^2\{1 - \cos(\lambda\eta_0)\}]}{\pi\lambda^2} d\lambda.$$

Hence if, as $\lambda \rightarrow 0$,

$$\mathbb{E}[\mathbf{U}_0^2\{1 - \cos(\lambda\eta_0)\}] \asymp |\lambda|^\alpha,$$

then (3) holds with

$$d = 1 - \alpha/2 \in (0, 1),$$

and hence X has memory parameter d .

It makes sense to apply the wavelet estimator of the memory parameter.

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Estimation result

The **bias term** behaves as in (6), if $\beta \leq 2 - \alpha$ and,

$$\mathbb{E}[\mathbf{U}_0^2 \mathbb{1}_{\eta_0 > t}] = c t^{-\alpha} (1 + O(|t|^{-\beta})), \quad (7)$$

$$\text{or } \mathbb{E}[\mathbf{U}_0^2 \{1 - \cos(\lambda \eta_0)\}] = c \lambda^\alpha (1 + O(|\lambda|^{-\beta})).$$

The **fluctuation term** behaves differently for $\alpha > 1$:

$$2^{-2dj} [\hat{\sigma}_j^2 - \text{Var}(W_{j,k})] = O_P(n_j^{-1/2} 2^{(\alpha-1)j/2})$$

(instead of $n_j^{-1/2}$ in the linear case).

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Rate of convergence

We obtain a rate of convergence **slower** than in the case of linear processes but:

If one observes the variables (\mathbf{U}_k, η_k) **directly**, the **best achievable rate** under the condition (7) is precisely the rate obtained by the wavelet estimator.

Hence we actually obtain the best achievable rate (recall that this is for $\beta \leq 2 - \alpha$).

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As a conclusion

Theorem A transformation of the data cannot increase the information.

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As a conclusion

Theorem A transformation of the data cannot increase the information.

A paradox Any estimation procedure is a transformation of the data.

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