Automata and expressions

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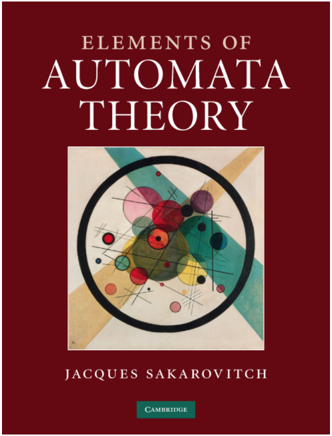
Based on



Chapter 1

HANDBOOK

OF

AUTOMATA THEORY

?

J.E. PIN, W. THOMAS, EDS.

Chapter 2

... much inspired by joint works with

Sylvain Lombardy (Univ. Bordeaux)

- *How expressions can code for automata?, RAIRO-TIA 2005, Corr. 2010*
- *The validity of weighted automata, CIAA 2012 & IJAC 2013*

and especially by the work on

- AWALI (formerly VAUCANSON, VAUCANSON2),
  a plateform for computing with weighted automata

Chapter I

A Platonic view of Kleene Theorem

Notation

- A *alphabet*, *i.e.* a finite set of *letters*

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- A^* set of *words*

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- $\mathrm{RegE}\, A^*$ set of *regular expressions* over A^*
  $\mathrm{Reg}\, A^*$ set of *regular languages* over A^*

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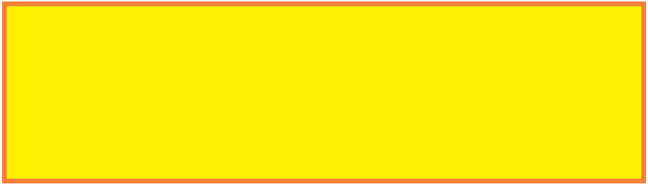
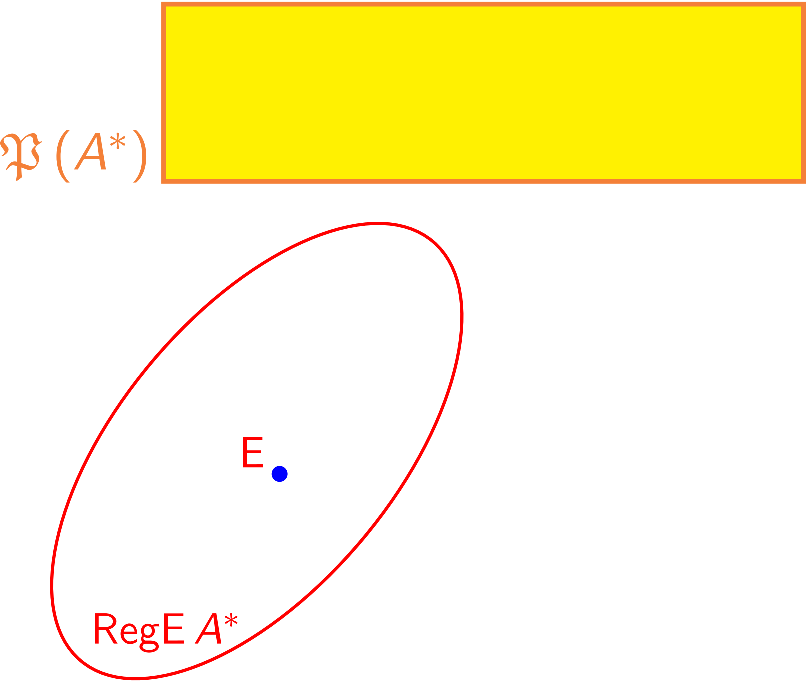
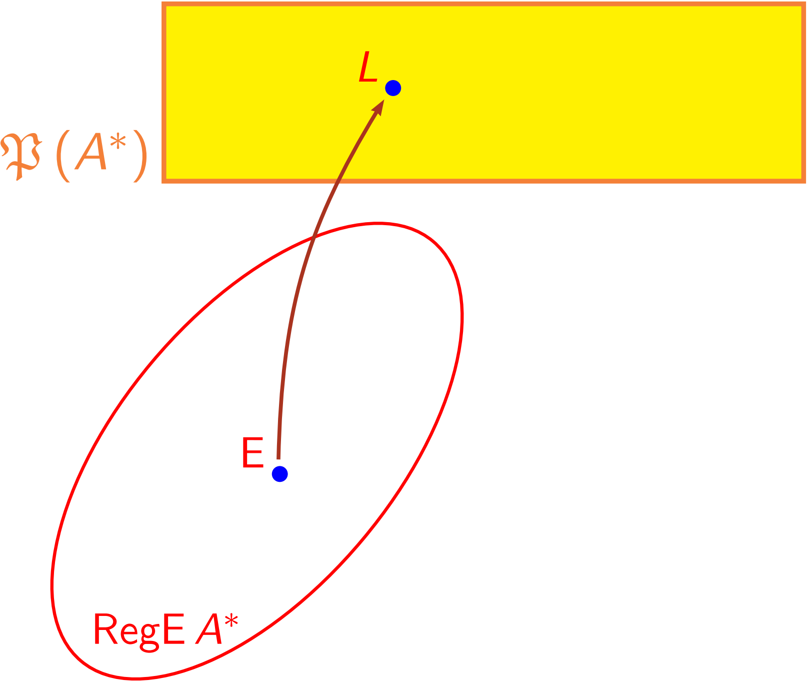
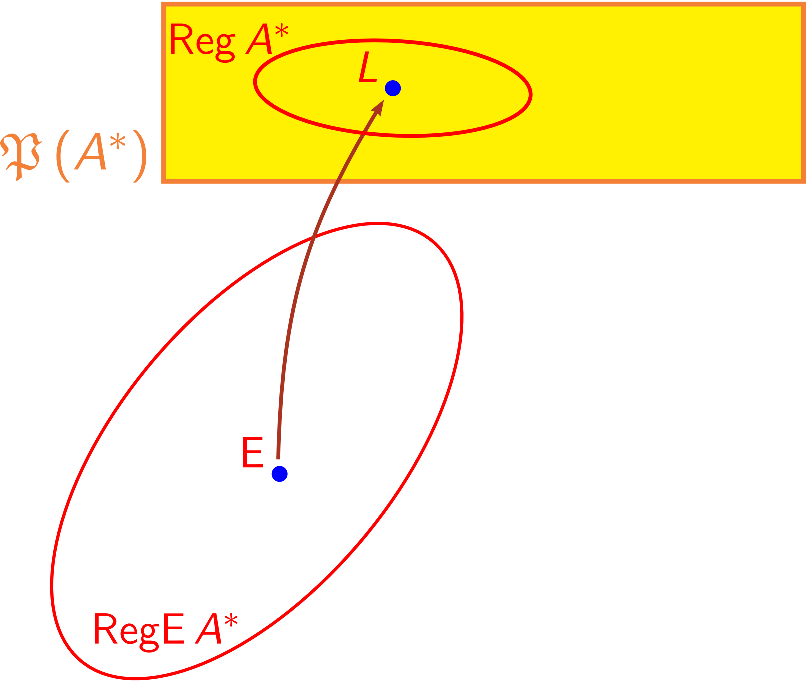
- A *alphabet*, *i.e.* a finite set of *letters*
- A^* set of *words*
- $L \subseteq A^*$ *language*
- $\text{RegE}\, A^*$ set of *regular expressions* over A^*
  $\text{Reg}\, A^*$ set of *regular languages* over A^*
- $\text{Aut}\, A^*$ set of *finite automata* over A^*
  $\text{Rec}\, A^*$ set of *recognizable languages* over A^*

$\mathfrak{P}(A^*)$

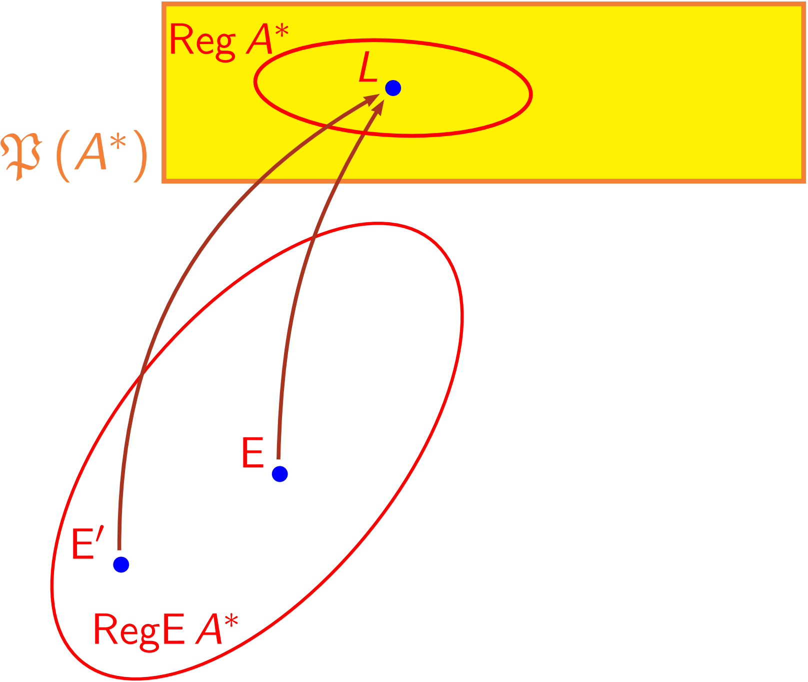



$\mathfrak{P}(A^*)$
E
RegE A^*

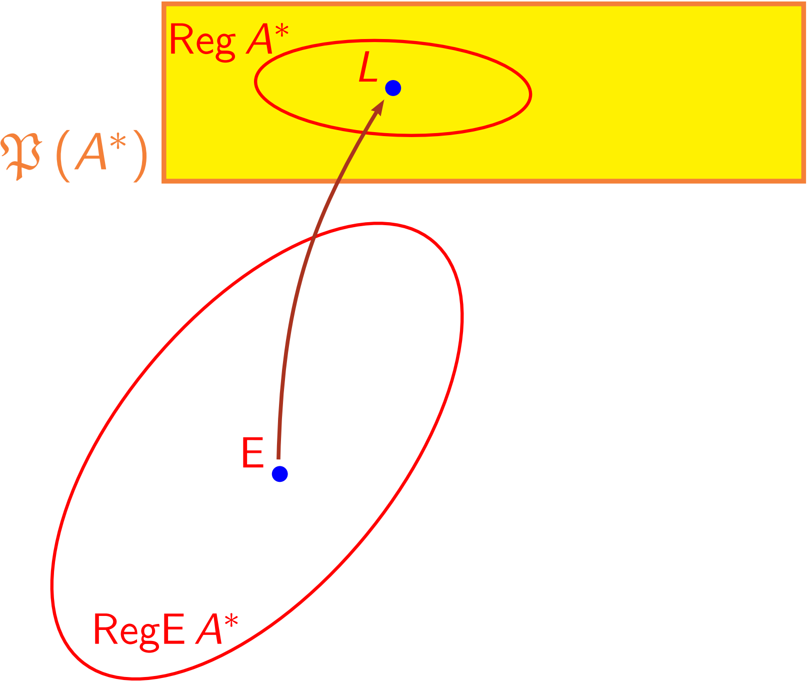

L
$\mathfrak{P}(A^*)$
E
RegE A^*



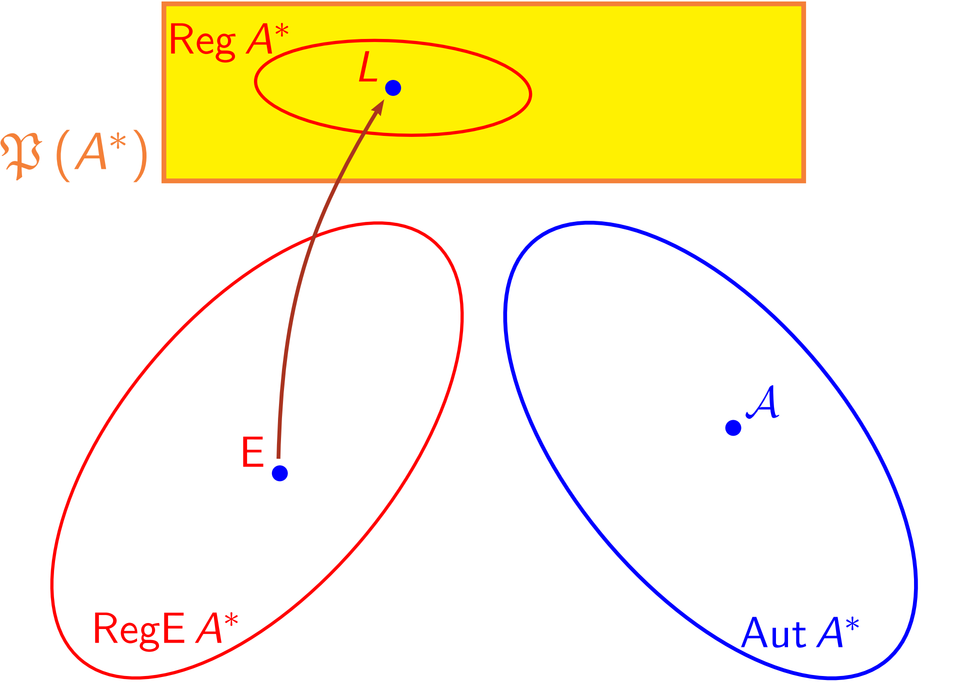
$\mathfrak{P}(A^*)$
Reg A^*
L
E
RegE A^*



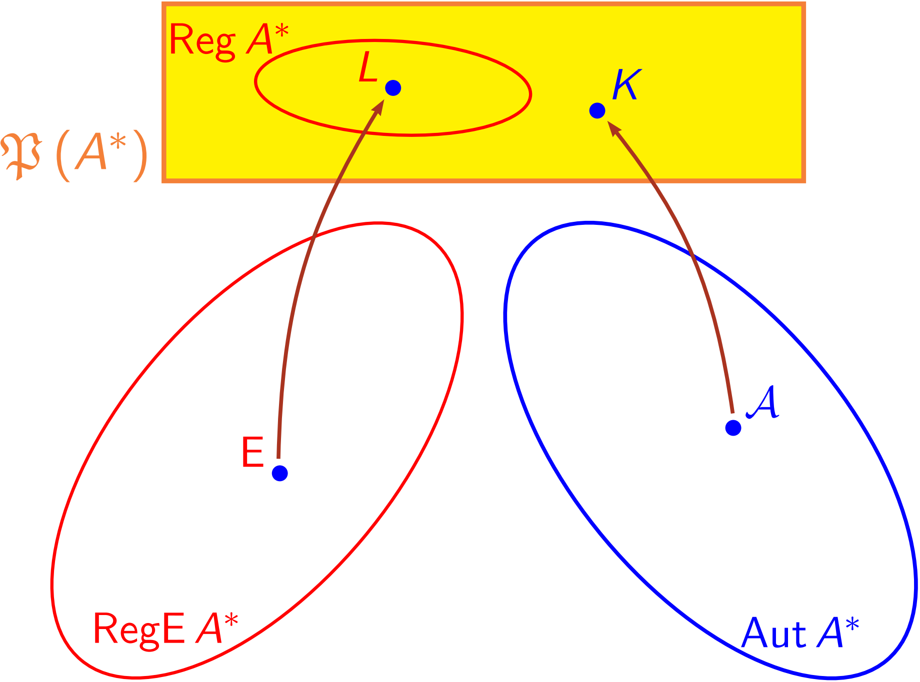
Reg A^*
L
$\mathfrak{P}(A^*)$
E
E′
RegE A^*

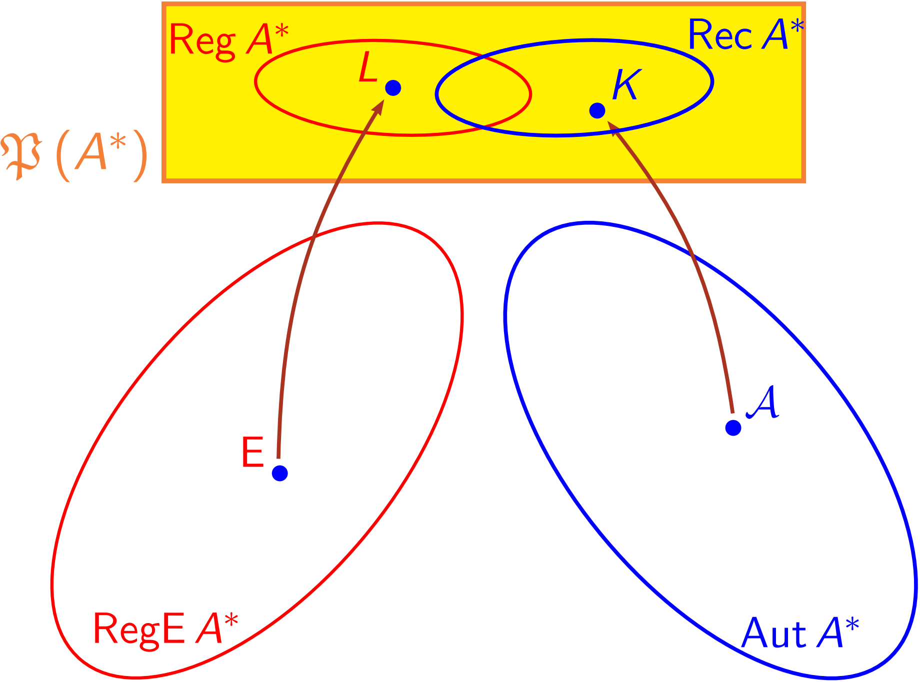


Reg A^*
L
$\mathfrak{P}(A^*)$
E
RegE A^*

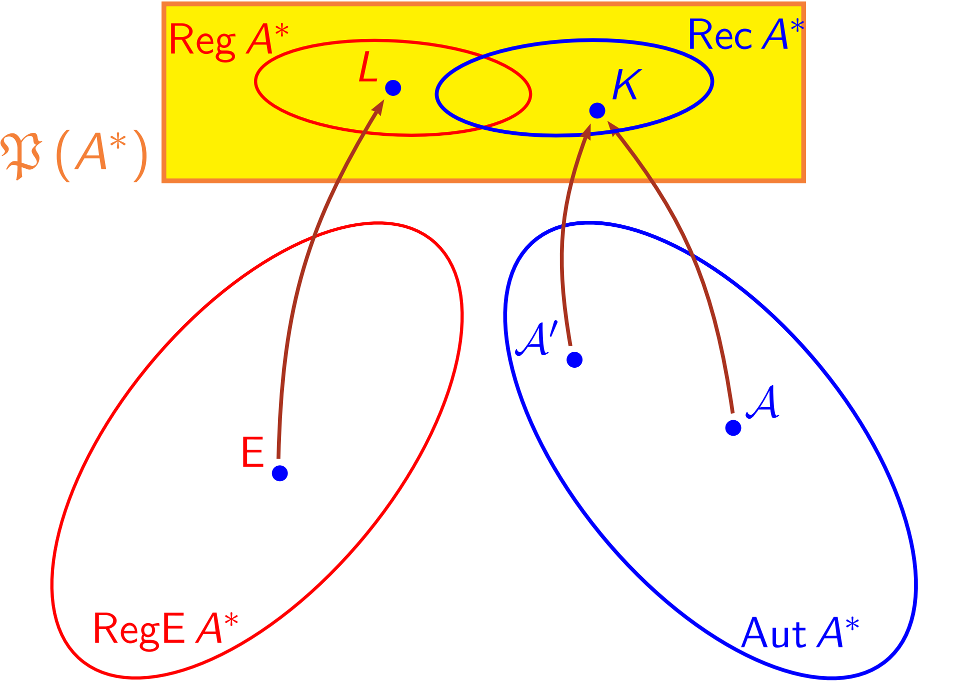


Reg A^*
L
$\mathfrak{P}(A^*)$
E
RegE A^*
$\mathcal{A}$
Aut A^*

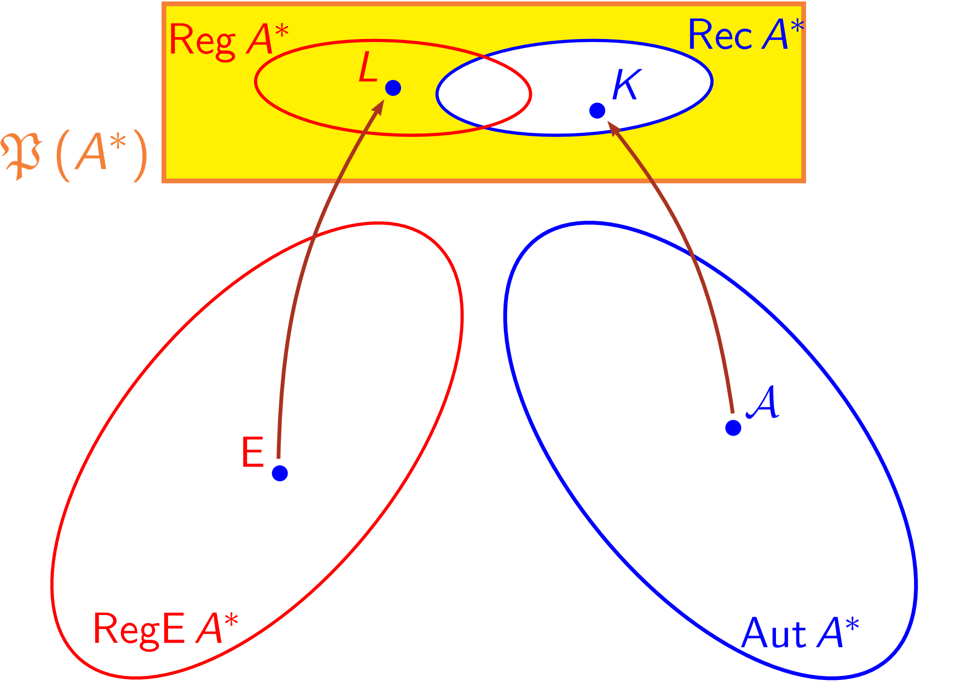

Reg A^*
L
K
$\mathfrak{P}(A^*)$
E
$\mathcal{A}$
RegE A^*
Aut A^*



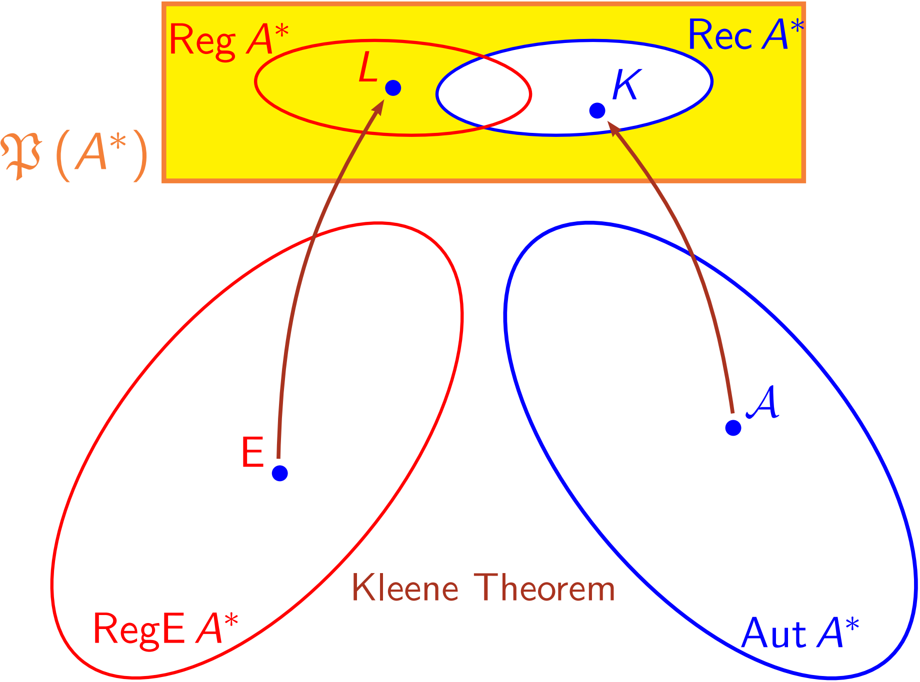
Reg A^*
Rec A^*
L
K
$\mathfrak{P}(A^*)$
E
$\mathcal{A}$
RegE A^*
Aut A^*

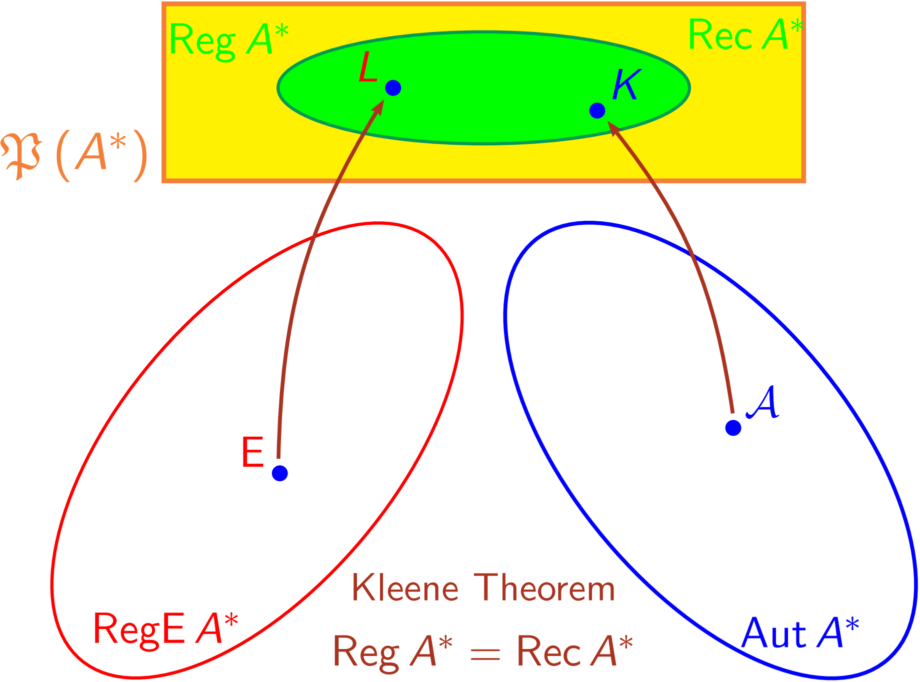


$\mathfrak{P}(A^*)$
Reg A^*
Rec A^*
L
K
$\mathcal{A}'$
$\mathcal{A}$
E
RegE A^*
Aut A^*

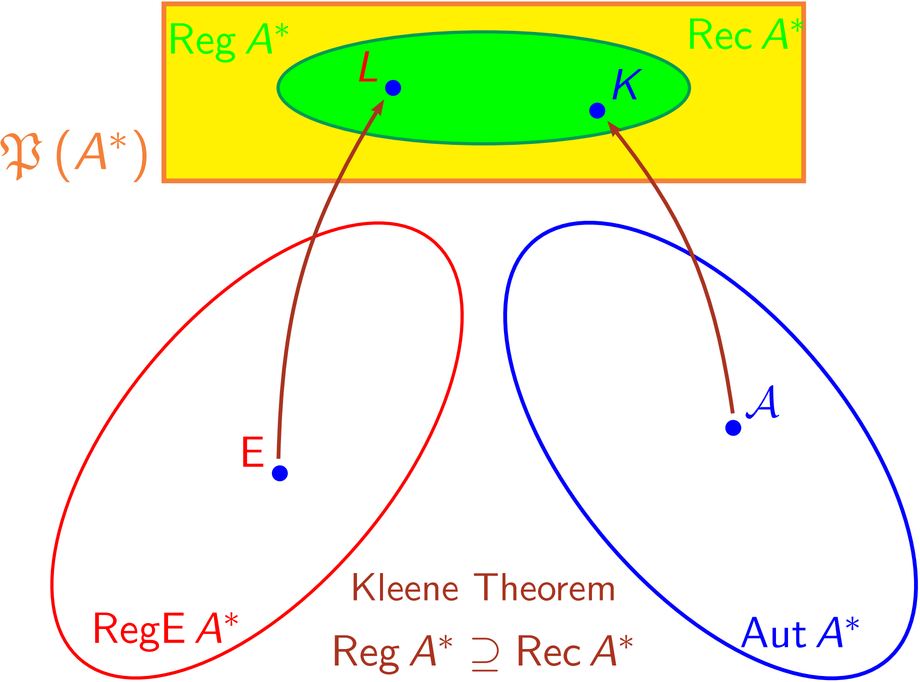


Reg A^*
Rec A^*
L
K
$\mathfrak{P}(A^*)$
E
$\mathcal{A}$
RegE A^*
Aut A^*

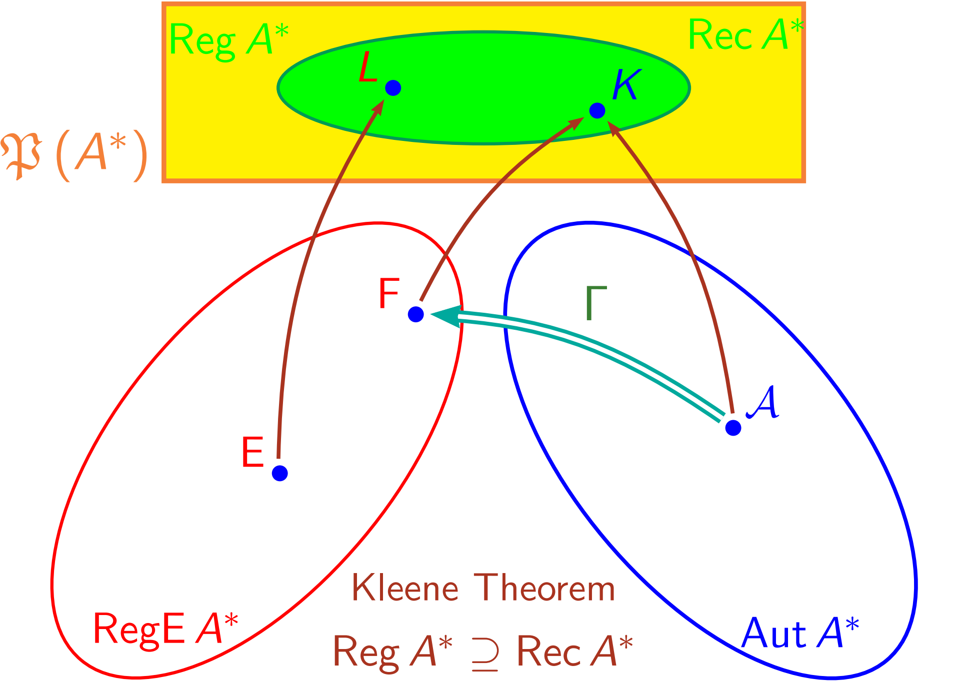

Reg A^*
Rec A^*
L
K
$\mathfrak{P}(A^*)$
E
$\mathcal{A}$
Kleene Theorem
RegE A^*
Aut A^*

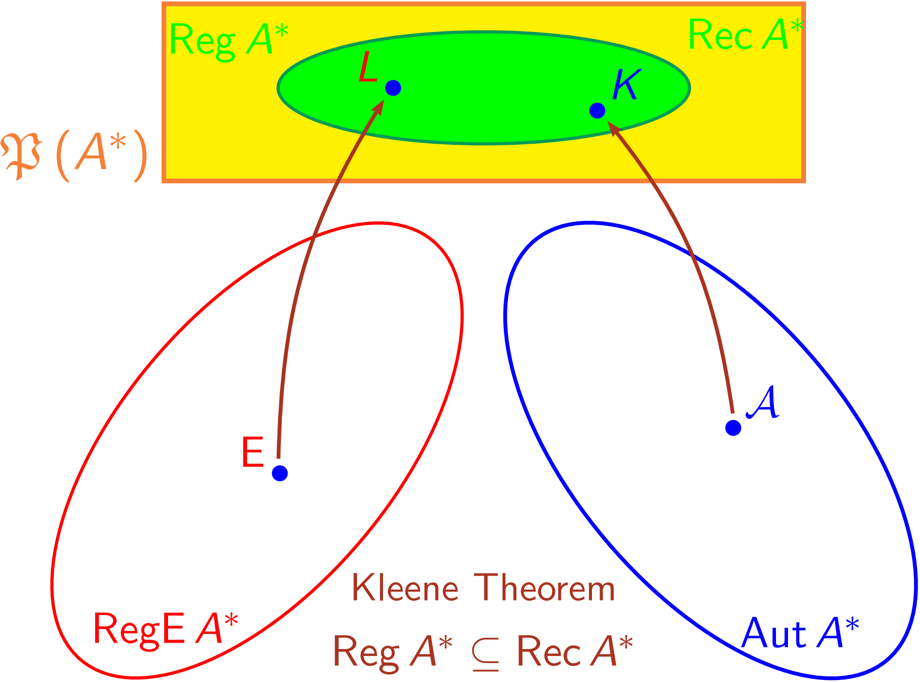


Reg A^*
Rec A^*
L
K
$\mathfrak{P}(A^*)$
E
$\mathcal{A}$
Kleene Theorem
RegE A^*
Reg A^* = Rec A^*
Aut A^*

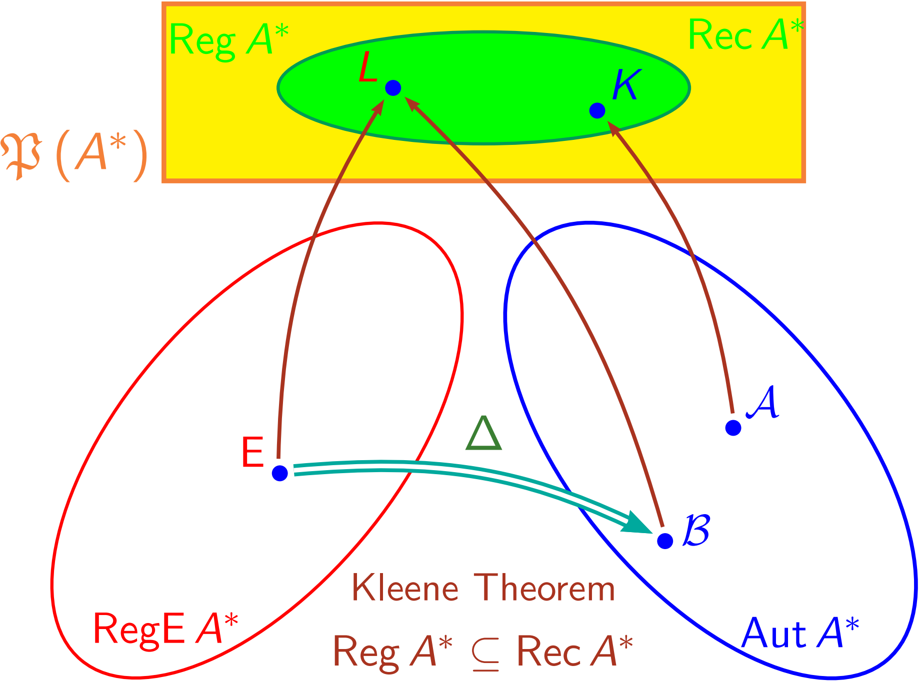


Reg A^*
Rec A^*
L
K
$\mathfrak{P}(A^*)$
E
$\mathcal{A}$
Kleene Theorem
RegE A^*
Aut A^*
Reg $A^* \supseteq$ Rec A^*

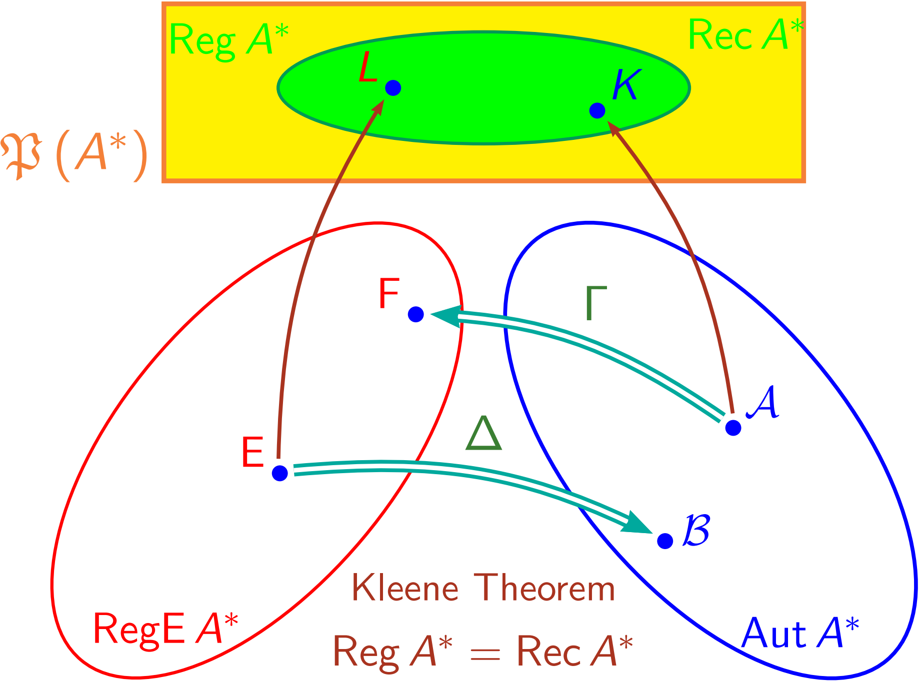

Reg A^*
Rec A^*
L
K
$\mathfrak{P}(A^*)$
F
Γ
$\mathcal{A}$
E
Kleene Theorem
RegE A^*
Reg $A^* \supseteq$ Rec A^*
Aut A^*



Reg A^*
Rec A^*
L
K
$\mathfrak{P}(A^*)$
E
$\mathcal{A}$
Kleene Theorem
RegE A^*
Reg $A^* \subseteq$ Rec A^*
Aut A^*



Reg A^*
Rec A^*
L
K
$\mathfrak{P}(A^*)$
Δ
E
$\mathcal{A}$
$\mathcal{B}$
Kleene Theorem
RegE A^*
Aut A^*
Reg $A^* \subseteq$ Rec A^*


Reg A^*
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Kleene Theorem
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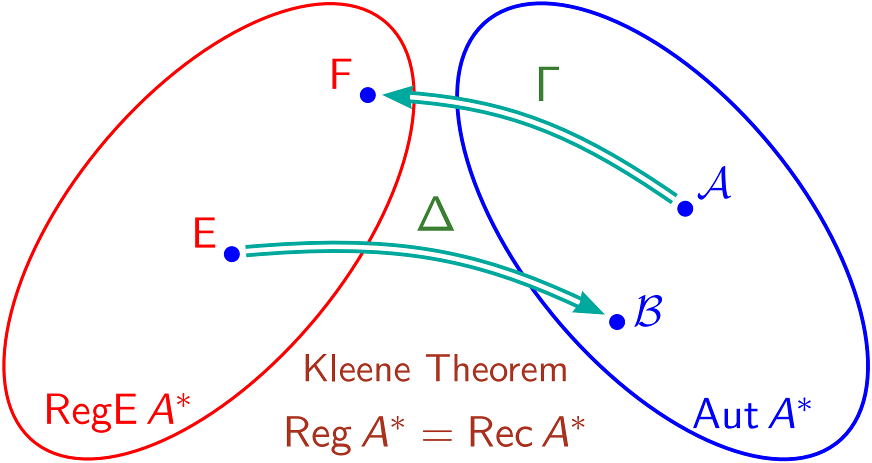
F
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RegE A^*
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Kleene Theorem
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Chapter II

From automata to expressions

The Γ algorithms

Computing an expression from an automaton

The Γ algorithms

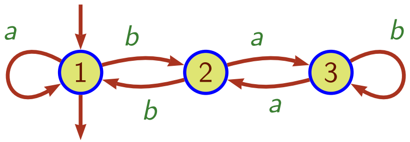
Computing an expression from an automaton

- Problem seen from a theoretical point of view
- Problem seen from an experimental point of view

The Γ algorithms

Computing an expression from an automaton

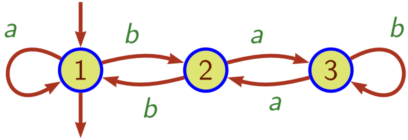
- Problem seen from a theoretical point of view
- Problem seen from an experimental point of view



The Γ algorithms

Computing an expression from an automaton

- Problem seen from a theoretical point of view
- Problem seen from an experimental point of view



$$\left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ b & 0 & a \\ 0 & a & b \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

The Γ algorithms

Computing an expression from an automaton

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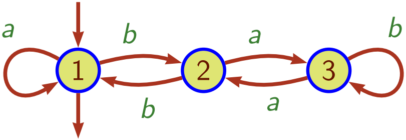


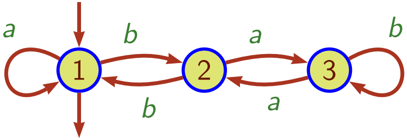
$$\left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ b & 0 & a \\ 0 & a & b \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$\mathcal{A} = \langle\, I, X, T \,\rangle \qquad L(\mathcal{A}) = I \cdot X^* \cdot T$$

The Γ algorithms

Computing an expression from an automaton

- Problem seen from a theoretical point of view
- Problem seen from an experimental point of view



$$\left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ b & 0 & a \\ 0 & a & b \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

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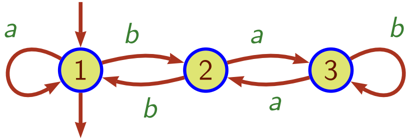
Computing the *star* of a matrix with entries in $\mathfrak{P}(A^*)$

The Γ algorithms

Computing an expression from an automaton

- Problem seen from a theoretical point of view
- Problem seen from an experimental point of view



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$$\mathcal{A} = \langle\, I, X, T \,\rangle \qquad L(\mathcal{A}) = I \cdot X^* \cdot T$$

Computing the *star* of a matrix with entries in $\mathfrak{P}(A^*)$

Computing the *quasi-inverse* of a matrix with entries in $\mathfrak{P}(A^*)$

The Γ algorithms

Theoretical point of view : methods of computations

The Γ algorithms

Theoretical point of view : methods of computations

1. Direct computation of the entries of X^* :

2. Computation of $X^* \cdot T$ as a fixed point:

3. Iterative computation of X^* :

4. Recursive computation of X^* :

The Γ algorithms

Theoretical point of view : methods of computations

1. Direct computation of the entries of X^* :
   state elimination method (Brzozowski–McCluskey)
2. Computation of $X^* \cdot T$ as a fixed point:
   solution of a system of linear equations
3. Iterative computation of X^* :
   McNaughton–Yamada algorithm
4. Recursive computation of X^* : Conway(?) algorithm

The Γ algorithms

Theoretical point of view : methods of computations

1. Direct computation of the entries of X^* :
   state elimination method (Brzozowski–McCluskey)

The Γ algorithms

Theoretical point of view : methods of computations

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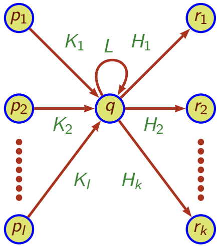

The Γ algorithms

Theoretical point of view : methods of computations

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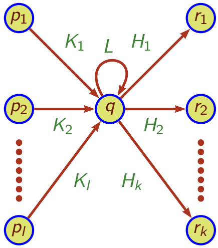

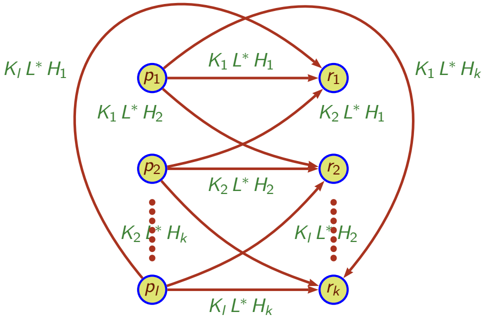

The Γ algorithms

Theoretical point of view : methods of computations

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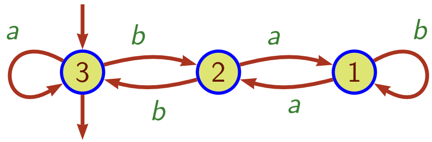

The Γ algorithms

Theoretical point of view : methods of computations

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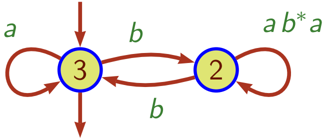

The Γ algorithms

Theoretical point of view : methods of computations

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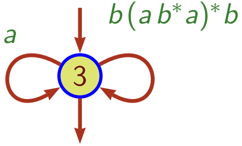

The Γ algorithms

Theoretical point of view : methods of computations

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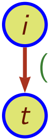

$(a + b\,(a\,b^*\,a)^*\,b)^*$

The Γ algorithms

Theoretical point of view : methods of computations

1. Direct computation of the entries of X^*
2. Computation of $X^* \cdot T$ as a fixed point
3. Iterative computation of X^*
4. Recursive computation of X^*

The Γ algorithms

Theoretical point of view : methods of computations

1. Direct computation of the entries of X^*
2. Computation of $X^* \cdot T$ as a fixed point
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Problem 1

Comparison between the expressions obtained with each method

The Γ algorithms

Theoretical point of view : methods of computations

1. Direct computation of the entries of X^*
2. Computation of $X^* \cdot T$ as a fixed point
3. Iterative computation of X^*
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Problem 1

Comparison between the expressions obtained with each method

For each method, the actual computation depends on an order on the set of states

The Γ algorithms

Theoretical point of view : methods of computations

1. Direct computation of the entries of X^*
2. Computation of $X^* \cdot T$ as a fixed point
3. Iterative computation of X^*
4. Recursive computation of X^*

Problem 1
Comparison between the expressions obtained with each method

For each method, the actual computation
depends on an order on the set of states

Problem 2
Comparison between the expressions obtained
in each method with distinct orders

The Γ algorithms

Theoretical point of view : methods of computations

1. Direct computation of the entries of X^*
2. Computation of $X^* \cdot T$ as a fixed point
3. Iterative computation of X^*
4. Recursive computation of X^*

Problem 1
Comparison between the expressions obtained with each method

For each method, the actual computation
depends on an order on the set of states

Problem 2
Comparison between the expressions obtained
in each method with distinct orders

Axiomatisation of regular expressions

Axiomatisation of regular expressions

Trivial and natural identities

$$\mathsf{E} + 0 \equiv 0 + \mathsf{E} \equiv \mathsf{E} \;\;, \quad \mathsf{E} \cdot 0 \equiv 0 \cdot \mathsf{E} \equiv 0 \;\;, \quad \mathsf{E} \cdot 1 \equiv 1 \cdot \mathsf{E} \equiv \mathsf{E} \qquad (\mathbf{T})$$

$$(\mathsf{E} + \mathsf{F}) + \mathsf{G} \equiv \mathsf{E} + (\mathsf{F} + \mathsf{G}) \;\;, \quad (\mathsf{E} \cdot \mathsf{F}) \cdot \mathsf{G} \equiv \mathsf{E} \cdot (\mathsf{F} \cdot \mathsf{G}) \qquad (\mathbf{A})$$

$$\mathsf{E} \cdot (\mathsf{F} + \mathsf{G}) \equiv \mathsf{E} \cdot \mathsf{F} + \mathsf{E} \cdot \mathsf{G} \;\;, \quad (\mathsf{E} + \mathsf{F}) \cdot \mathsf{G} \equiv \mathsf{E} \cdot \mathsf{G} + \mathsf{F} \cdot \mathsf{G} \qquad (\mathbf{D})$$

$$\mathsf{E} + \mathsf{F} \;\equiv\; \mathsf{F} + \mathsf{E} \qquad (\mathbf{C})$$

Axiomatisation of regular expressions

Trivial and natural identities

$$E + 0 \equiv 0 + E \equiv E \;\;,\quad E \cdot 0 \equiv 0 \cdot E \equiv 0 \;\;,\quad E \cdot 1 \equiv 1 \cdot E \equiv E \qquad (\mathbf{T})$$

$$(E + F) + G \equiv E + (F + G) \;\;,\quad (E \cdot F) \cdot G \equiv E \cdot (F \cdot G) \qquad (\mathbf{A})$$

$$E \cdot (F + G) \equiv E \cdot F + E \cdot G \;\;,\quad (E + F) \cdot G \equiv E \cdot G + F \cdot G \qquad (\mathbf{D})$$

$$E + F \equiv F + E \qquad (\mathbf{C})$$

Aperiodic identities.

$$E^* \equiv 1 + E \cdot E^* \;\;,\quad E^* \equiv 1 + E^* \cdot E \qquad (\mathbf{U})$$

$$(E + F)^* \equiv E^* \cdot (F \cdot E^*)^* \;\;,\quad (E + F)^* \equiv (E^* \cdot F)^* \cdot E^* \qquad (\mathbf{S})$$

$$(E \cdot F)^* \equiv 1 + E \cdot (F \cdot E)^* \cdot F \qquad (\mathbf{P})$$

Axiomatisation of regular expressions

Trivial and natural identities

$$\mathsf{E} + 0 \equiv 0 + \mathsf{E} \equiv \mathsf{E} \;\;,\quad \mathsf{E} \cdot 0 \equiv 0 \cdot \mathsf{E} \equiv 0 \;\;,\quad \mathsf{E} \cdot 1 \equiv 1 \cdot \mathsf{E} \equiv \mathsf{E} \qquad (\mathbf{T})$$

$$(\mathsf{E} + \mathsf{F}) + \mathsf{G} \equiv \mathsf{E} + (\mathsf{F} + \mathsf{G}) \;\;,\quad (\mathsf{E} \cdot \mathsf{F}) \cdot \mathsf{G} \equiv \mathsf{E} \cdot (\mathsf{F} \cdot \mathsf{G}) \qquad (\mathbf{A})$$

$$\mathsf{E} \cdot (\mathsf{F} + \mathsf{G}) \equiv \mathsf{E} \cdot \mathsf{F} + \mathsf{E} \cdot \mathsf{G} \;\;,\quad (\mathsf{E} + \mathsf{F}) \cdot \mathsf{G} \equiv \mathsf{E} \cdot \mathsf{G} + \mathsf{F} \cdot \mathsf{G} \qquad (\mathbf{D})$$

$$\mathsf{E} + \mathsf{F} \equiv \mathsf{F} + \mathsf{E} \qquad (\mathbf{C})$$

Aperiodic identities.

$$\mathsf{E}^* \equiv 1 + \mathsf{E} \cdot \mathsf{E}^* \;\;,\quad \mathsf{E}^* \equiv 1 + \mathsf{E}^* \cdot \mathsf{E} \qquad (\mathbf{U})$$

$$(\mathsf{E} + \mathsf{F})^* \equiv \mathsf{E}^* \cdot (\mathsf{F} \cdot \mathsf{E}^*)^* \;\;,\quad (\mathsf{E} + \mathsf{F})^* \equiv (\mathsf{E}^* \cdot \mathsf{F})^* \cdot \mathsf{E}^* \qquad (\mathbf{S})$$

$$(\mathsf{E} \cdot \mathsf{F})^* \equiv 1 + \mathsf{E} \cdot (\mathsf{F} \cdot \mathsf{E})^* \cdot \mathsf{F} \qquad (\mathbf{P})$$

Cyclic identities.

$$\mathsf{E}^* \equiv \mathsf{E}^{<\mathsf{n}} \cdot (\mathsf{E}^{\mathsf{n}})^* \qquad (\mathbf{Z}_n)$$

Axiomatisation of regular expressions

Trivial and natural identities

$$\mathsf{E} + 0 \equiv 0 + \mathsf{E} \equiv \mathsf{E} \;,\quad \mathsf{E} \cdot 0 \equiv 0 \cdot \mathsf{E} \equiv 0 \;,\quad \mathsf{E} \cdot 1 \equiv 1 \cdot \mathsf{E} \equiv \mathsf{E} \qquad (\mathbf{T})$$

$$(\mathsf{E} + \mathsf{F}) + \mathsf{G} \equiv \mathsf{E} + (\mathsf{F} + \mathsf{G}) \;,\quad (\mathsf{E} \cdot \mathsf{F}) \cdot \mathsf{G} \equiv \mathsf{E} \cdot (\mathsf{F} \cdot \mathsf{G}) \qquad (\mathbf{A})$$

$$\mathsf{E} \cdot (\mathsf{F} + \mathsf{G}) \equiv \mathsf{E} \cdot \mathsf{F} + \mathsf{E} \cdot \mathsf{G} \;,\quad (\mathsf{E} + \mathsf{F}) \cdot \mathsf{G} \equiv \mathsf{E} \cdot \mathsf{G} + \mathsf{F} \cdot \mathsf{G} \qquad (\mathbf{D})$$

$$\mathsf{E} + \mathsf{F} \equiv \mathsf{F} + \mathsf{E} \qquad (\mathbf{C})$$

Aperiodic identities.

$$\mathsf{E}^* \equiv 1 + \mathsf{E} \cdot \mathsf{E}^* \;,\quad \mathsf{E}^* \equiv 1 + \mathsf{E}^* \cdot \mathsf{E} \qquad (\mathbf{U})$$

$$(\mathsf{E} + \mathsf{F})^* \equiv \mathsf{E}^* \cdot (\mathsf{F} \cdot \mathsf{E}^*)^* \;,\quad (\mathsf{E} + \mathsf{F})^* \equiv (\mathsf{E}^* \cdot \mathsf{F})^* \cdot \mathsf{E}^* \qquad (\mathbf{S})$$

$$(\mathsf{E} \cdot \mathsf{F})^* \equiv 1 + \mathsf{E} \cdot (\mathsf{F} \cdot \mathsf{E})^* \cdot \mathsf{F} \qquad (\mathbf{P})$$

Cyclic identities.

$$\mathsf{E}^* \equiv \mathsf{E}^{<\mathsf{n}} \cdot (\mathsf{E}^{\mathsf{n}})^* \qquad (\mathbf{Z}_n)$$

Idempotency identities.

$$\mathsf{E} + \mathsf{E} \equiv \mathsf{E} \qquad (\mathbf{I})$$

$$(\mathsf{E}^*)^* \equiv \mathsf{E}^* \qquad (\mathbf{J})$$

The Γ algorithms

$$\mathcal{A} = \langle\, Q, A, X, I, T \,\rangle$$

1. State elimination method $\mathsf{E}_{\mathcal{A}}$
2. Solution of a system of linear equations $\mathsf{S}_{\mathcal{A}}$
3. McNaughton–Yamada algorithm $\mathsf{M}_{\mathcal{A}}$
4. Recursive computation of X^* $\mathsf{C}_{\mathcal{A}}$

The Γ algorithms

$\mathcal{A} = \langle\, Q, A, X, I, T \,\rangle$ $\qquad$ ω ordering on Q

1. State elimination method $\qquad$ $\mathsf{E}(\omega)$
2. Solution of a system of linear equations $\qquad$ $\mathsf{S}(\omega)$
3. McNaughton–Yamada algorithm $\qquad$ $\mathsf{M}(\omega)$
4. Recursive computation of X^* $\qquad$ $\mathsf{C}(\omega)$

The Γ algorithms

$\mathcal{A} = \langle\, Q, A, X, \{p\}, \{q\}\,\rangle$ $\qquad$ ω ordering on Q

1. State elimination method $\qquad$ $\mathsf{E}(\omega, p, q)$
2. Solution of a system of linear equations $\qquad$ $[\mathsf{S}(\omega, q)]_p$
3. McNaughton–Yamada algorithm $\qquad$ $[\mathsf{M}(\omega)]_{p,q}$
4. Recursive computation of X^* $\qquad$ $[\mathsf{C}(\omega)]_{p,q}$

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4. Recursive computation of X^* $[\mathsf{C}(\omega)]_{p,q}$

Proposition

$$[\mathsf{S}(\omega, q)]_p \;=\; \mathsf{E}(\omega, p, q)$$

The Γ algorithms

$$\mathcal{A} = \langle\, Q, A, X, \{p\}, \{q\} \,\rangle \qquad \omega \text{ ordering on } Q$$

1. State elimination method $\qquad \mathsf{E}(\omega, p, q)$
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Proposition (S. 03)

$$\mathbf{U} \vdash \qquad [\mathsf{M}(\omega)]_{p,q} \quad \equiv \quad \mathsf{E}(\omega, p, q)$$

The Γ algorithms

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Theorem (Conway 71, Krob 92, S.03)

For any two orderings ω and ω' on Q

$$\mathbf{S} \wedge \mathbf{P} \quad \vdash \quad \mathsf{E}(\omega, p, q) \quad \equiv \quad \mathsf{E}(\omega', p, q)$$

The Γ algorithms

$\mathcal{A} = \langle\, Q, A, X, \{p\}, \{q\}\,\rangle$ $\qquad$ ω ordering on Q

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4. Recursive computation of X^* $\qquad$ $[\mathsf{C}(\omega)]_{p,q}$

Conjecture

For any ordering ω *on* Q

there exists an ordering ω' *such that*

$$\mathbf{U} \vdash \qquad [\mathsf{C}(\omega)]_{p,q} \quad \equiv \quad \mathsf{E}(\omega', p, q)$$

The Γ algorithms

$\mathcal{A} = \langle\, Q, A, X, \{p\}, \{q\}\,\rangle$ ω ordering on Q

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4. Recursive computation of X^* $[\mathsf{C}(\omega)]_{p,q}$

Conjecture

For any recursive division τ of Q
there exists an ordering ω' such that

$$\mathbf{U} \vdash \qquad [\mathsf{C}(\tau)]_{p,q} \quad \equiv \quad \mathsf{E}(\omega', p, q)$$

The Γ algorithms

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Conclusion

Several algorithms, essentially ONE result

(from a theoretical point of view)

The Γ algorithms: an experimental point of view

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The size of E computed from $\mathcal{A}$
may be exponential in the number of states of $\mathcal{A}$

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The Γ algorithms: an experimental point of view

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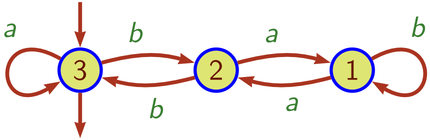

The Γ algorithms: an experimental point of view

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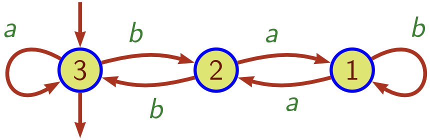

$$E_2 = (a + b(ab^*a)^*b)^*$$

The Γ algorithms: an experimental point of view

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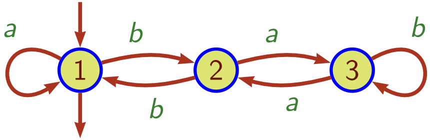

$$E_2 = (a + b(ab^*a)^*b)^*$$

$$E_1 = a^* + a^*b(ba^*b)^*ba^* + a^*b(ba^*b)^*a(b + a(ba^*b)^*a)^*a(ba^*b)^*ba^*$$

The Γ algorithms: an experimental point of view

Heuristics for the ordering of states proves to be (very) useful.

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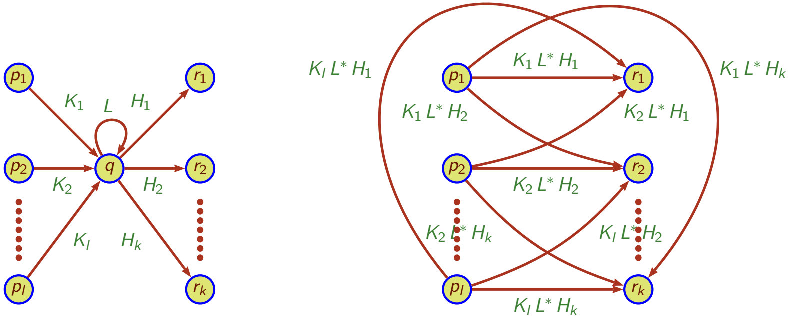

The Γ algorithms: an experimental point of view

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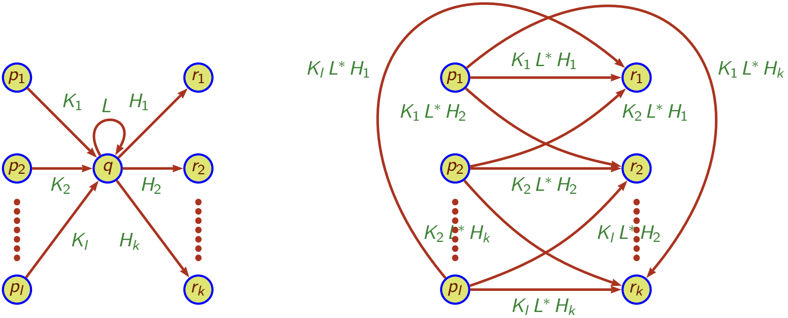
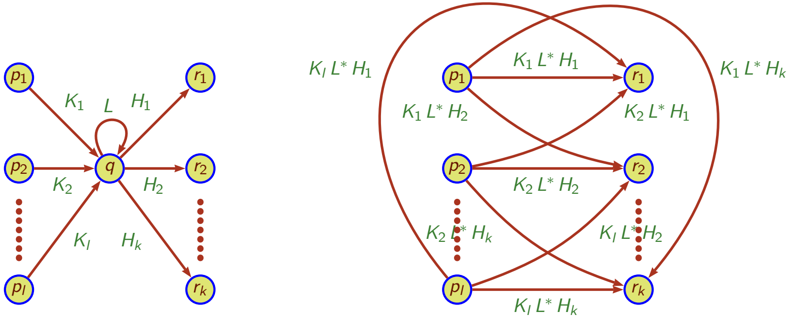


- The naive heuristic

The Γ algorithms: an experimental point of view

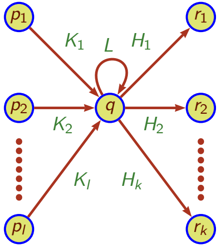
Heuristics for the ordering of states proves to be (very) useful.

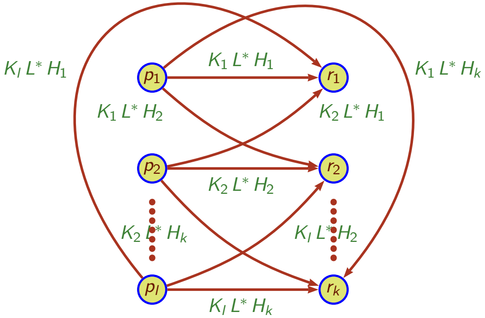


- The naive heuristic
- The Delgado–Morais heuristic (CIAA 04)

The Γ algorithms: an experimental point of view

Heuristics for the ordering of states proves to be (very) useful.





- The naive heuristic
- The Delgado–Morais heuristic (CIAA 04)

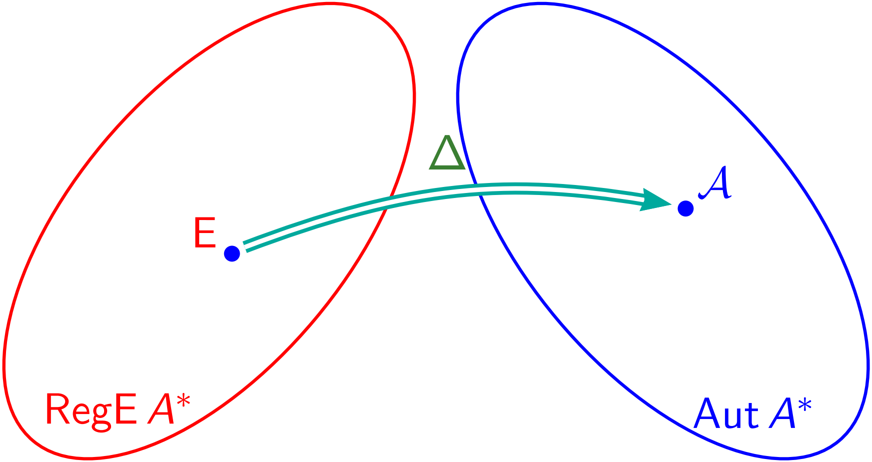
The proof of the heuristic is in the computing.

Chapter III

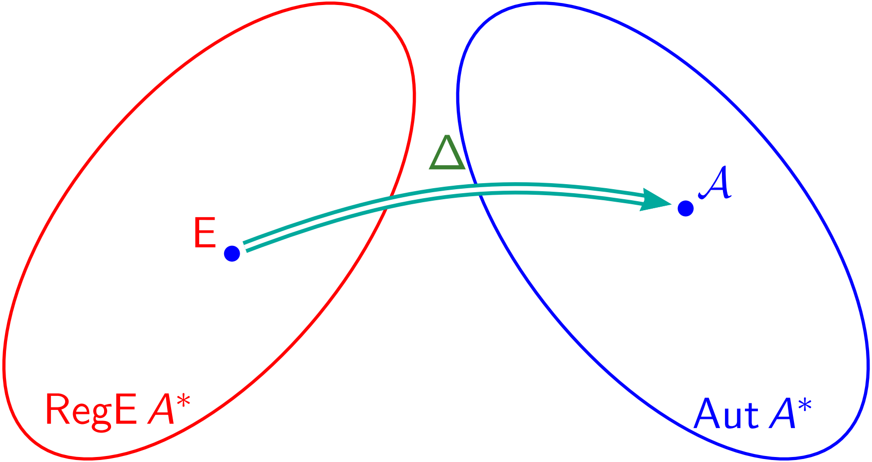
From expressions to automata

The Δ algorithms

Computing an automaton from an expression

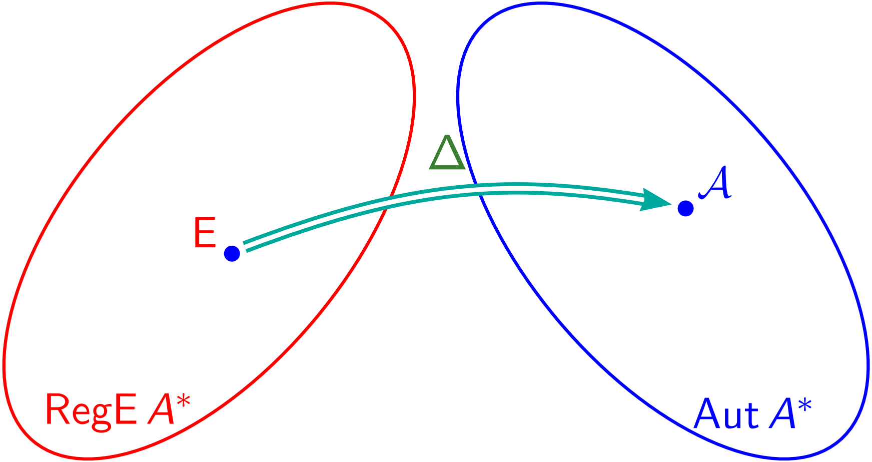


The Δ algorithms



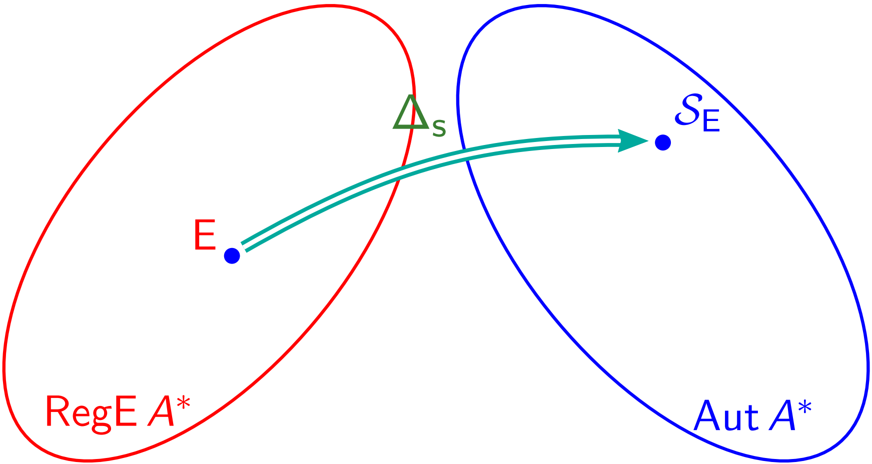
The Δ algorithms

- Standard automaton of E
- Derived term automaton of E



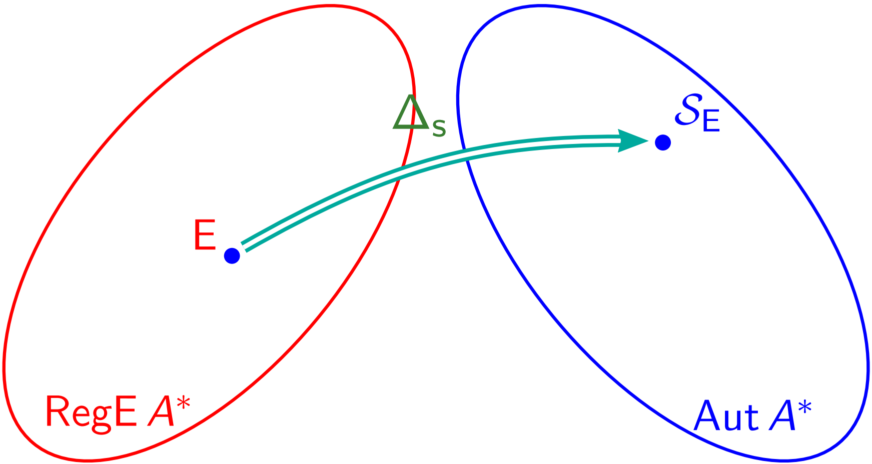
The Δ algorithms

- Standard automaton of E — position, Glushkov
- Derived term automaton of E



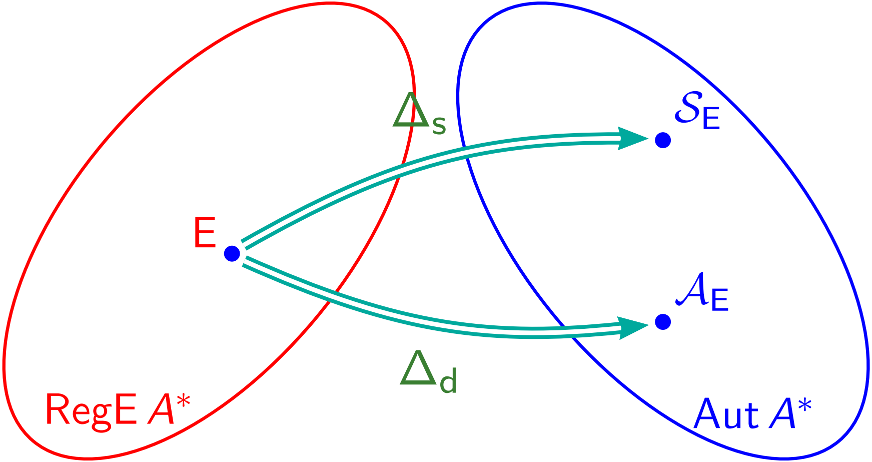
The Δ algorithms

- Standard automaton of E — Thompson + closure
- Derived term automaton of E



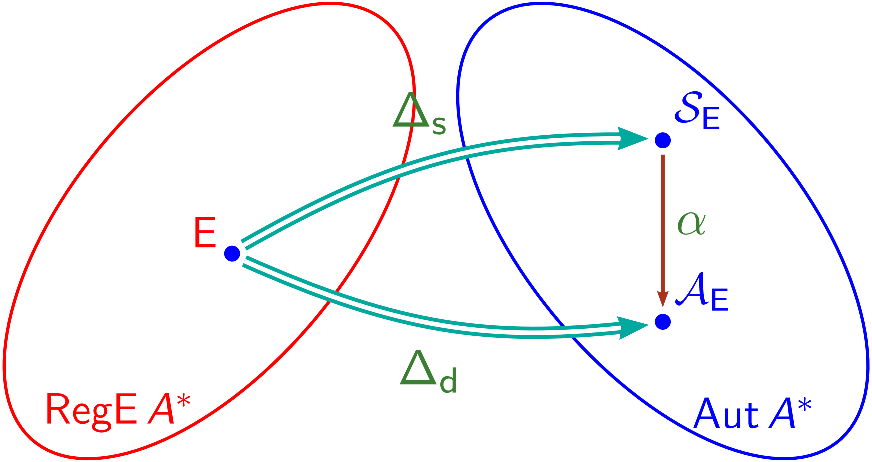
The Δ algorithms

- Standard automaton of E — position, Glushkov
- Derived term automaton of E — Brzozowski–Antimirov



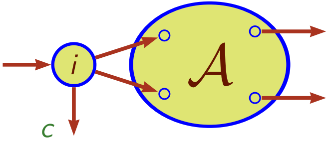
The Δ algorithms

- Standard automaton of E — position, Glushkov
- Derived term automaton of E — Brzozowski–Antimirov



The standard automaton of an expression

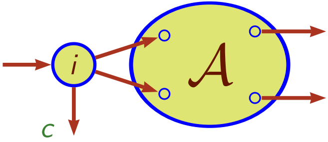
Definition of a standard automaton



$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & \boxed{\;0\;} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{\;J\;} \\ \boxed{0} & \boxed{\;F\;} \end{pmatrix}, \begin{pmatrix} c \\ \boxed{U} \end{pmatrix} \right\rangle$$

The standard automaton of an expression

Definition of a standard automaton



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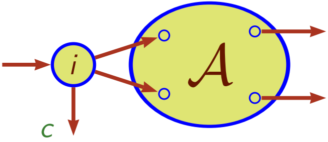
Operations on standard automata

$\mathcal{A} + \mathcal{B}$ $\qquad$ $\mathcal{A} \cdot \mathcal{B}$ $\qquad$ $\mathcal{A}^*$

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} \\ \boxed{0} & \boxed{F} \end{pmatrix}, \begin{pmatrix} c \\ \boxed{U} \end{pmatrix} \right\rangle$$

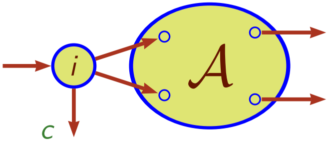
Operations on standard automata

$\mathcal{A} + \mathcal{B}$ $\qquad$ $\mathcal{A} \cdot \mathcal{B}$ $\qquad$ $\mathcal{A}^*$

$$\mathcal{A} + \mathcal{B} = \left\langle \begin{pmatrix} 1 & \boxed{0} & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} & \boxed{K} \\ \boxed{0} & \boxed{F} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{G} \end{pmatrix}, \begin{pmatrix} c + d \\ \boxed{U} \\ \boxed{V} \end{pmatrix} \right\rangle$$

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle$$

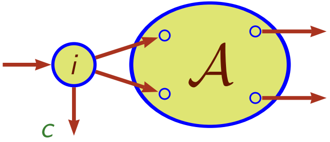
Operations on standard automata

$\mathcal{A} + \mathcal{B}$ $\qquad$ $\mathcal{A} \cdot \mathcal{B}$ $\qquad$ $\mathcal{A}^*$

$$\mathcal{A} \cdot \mathcal{B} = \left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J & c\,K \\ 0 & F & U \cdot K \\ 0 & 0 & G \end{pmatrix}, \begin{pmatrix} c\,d \\ Ud \\ V \end{pmatrix} \right\rangle$$

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & \boxed{\;0\;} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{\;J\;} \\ \boxed{0} & \boxed{\;F\;} \end{pmatrix}, \begin{pmatrix} c \\ \boxed{U} \end{pmatrix} \right\rangle$$

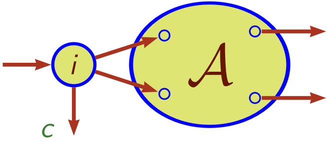
Operations on standard automata

$$\mathcal{A} + \mathcal{B} \qquad\qquad \mathcal{A} \cdot \mathcal{B} \qquad\qquad \mathcal{A}^*$$

$$\mathcal{A}^* = \left\langle \begin{pmatrix} 1 & \boxed{\;0\;} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{\;J\;} \\ \boxed{0} & \boxed{\;H\;} \end{pmatrix}, \begin{pmatrix} 1 \\ \boxed{U} \end{pmatrix} \right\rangle \qquad \text{with} \quad H = U \cdot J + F$$

The standard automaton of an expression

Definition of a standard automaton



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Operations on standard automata

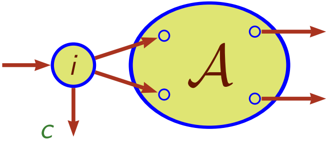
$\mathcal{A} + \mathcal{B}$ $\qquad$ $\mathcal{A} \cdot \mathcal{B}$ $\qquad$ $\mathcal{A}^*$

Definition of Δ_{d}

Recursive application of the operations $\qquad \Delta_{\mathsf{d}}(\mathsf{E}) = \mathcal{S}_{\mathsf{E}}$

The standard automaton of an expression

Definition of a standard automaton



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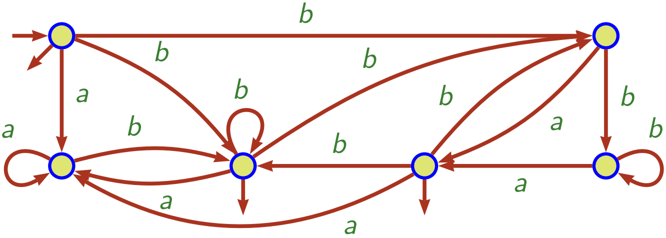
Operations on standard automata

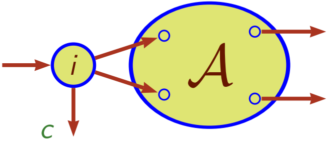
$\mathcal{A} + \mathcal{B}$ $\qquad$ $\mathcal{A} \cdot \mathcal{B}$ $\qquad$ $\mathcal{A}^*$

Example $\mathsf{E}_1 = (a^* b + b\, b^* a)^*$



The standard automaton of an expression

Definition of a standard automaton



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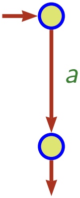
Operations on standard automata

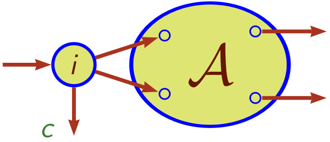
$$\mathcal{A} + \mathcal{B} \qquad \mathcal{A} \cdot \mathcal{B} \qquad \mathcal{A}^*$$

Example $\mathsf{E}_1 = a$



The standard automaton of an expression

Definition of a standard automaton

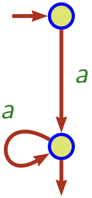


$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle$$

Operations on standard automata

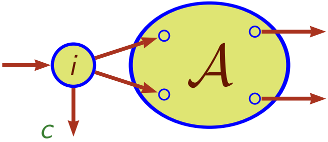
$\mathcal{A} + \mathcal{B}$ $\qquad$ $\mathcal{A} \cdot \mathcal{B}$ $\qquad$ $\mathcal{A}^*$

Example $\mathsf{E}_1 = a^*$



The standard automaton of an expression

Definition of a standard automaton

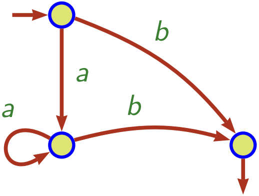


$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle$$

Operations on standard automata

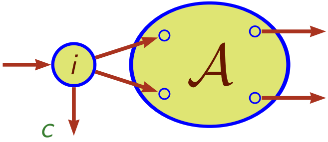
$\mathcal{A} + \mathcal{B}$ $\qquad$ $\mathcal{A} \cdot \mathcal{B}$ $\qquad$ $\mathcal{A}^*$

Example $\mathsf{E}_1 = a^* b$



The standard automaton of an expression

Definition of a standard automaton

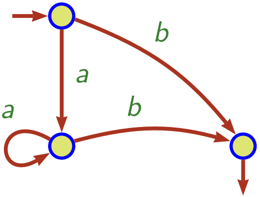


$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & \boxed{\;0\;} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{\;J\;} \\ \boxed{0} & \boxed{\;F\;} \end{pmatrix}, \begin{pmatrix} c \\ \boxed{U} \end{pmatrix} \right\rangle$$

Operations on standard automata

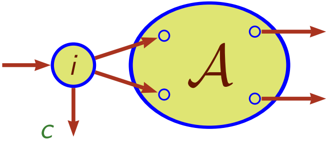
$\mathcal{A} + \mathcal{B}$ $\qquad$ $\mathcal{A} \cdot \mathcal{B}$ $\qquad$ $\mathcal{A}^*$

Example $\mathsf{E}_1 = a^* b$ $\quad$ b



The standard automaton of an expression

Definition of a standard automaton

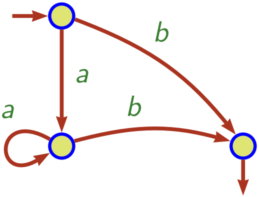


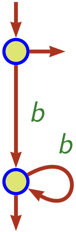
$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} \\ \boxed{0} & \boxed{F} \end{pmatrix}, \begin{pmatrix} c \\ \boxed{U} \end{pmatrix} \right\rangle$$

Operations on standard automata

$$\mathcal{A} + \mathcal{B} \qquad\qquad \mathcal{A} \cdot \mathcal{B} \qquad\qquad \mathcal{A}^*$$

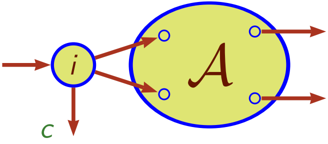
Example $\mathsf{E}_1 = a^* b \qquad b^*$





The standard automaton of an expression

Definition of a standard automaton



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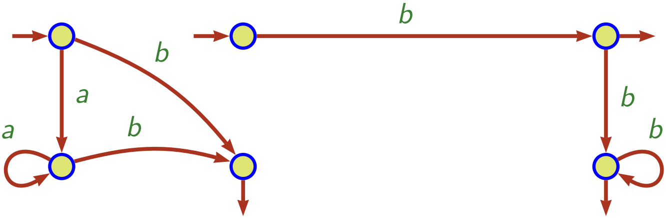

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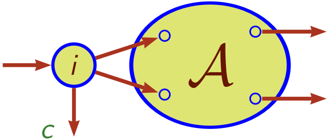

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Operations on standard automata

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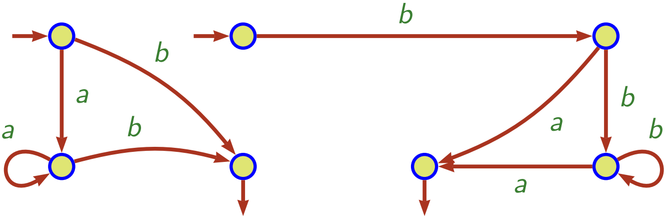

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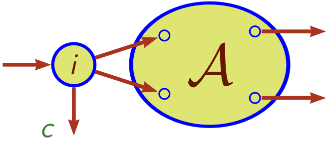

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Operations on standard automata

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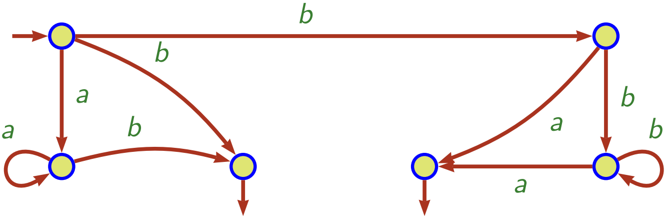
Example $\mathsf{E}_1 = a^* b + b\, b^* a$



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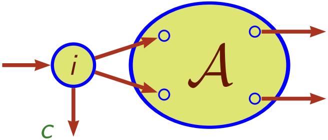

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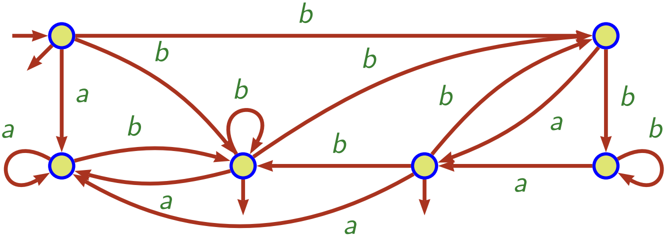
Operations on standard automata

$$\mathcal{A} + \mathcal{B} \qquad \mathcal{A} \cdot \mathcal{B} \qquad \mathcal{A}^*$$

Example $\mathsf{E}_1 = (a^* b + b\, b^* a)^*$



The standard automaton of an expression

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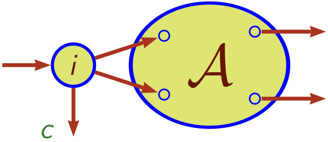

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Proposition

Size of $\mathcal{S}_{\mathsf{E}}$ *is* $\ell(\mathsf{E}) + 1$

Proposition

The complexity of Δ_{d} *is cubic*

The standard automaton of an expression

Definition (Brüggemann-Klein 92)

E is *in star-normal form* (SNF) if and only if
for any F such that F^* is a subexpression of E, $\mathsf{c}(\mathsf{F}) = 0$

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Theorem (B-K 92)

For any E *, an* $\mathsf{E}^\bullet$ *can be computed in linear time, s.t.*

(i) $\mathsf{E}^\bullet$ *is in star-normal form*

(ii) $\mathcal{S}_{\mathsf{E}^\bullet} = \mathcal{S}_{\mathsf{E}}$

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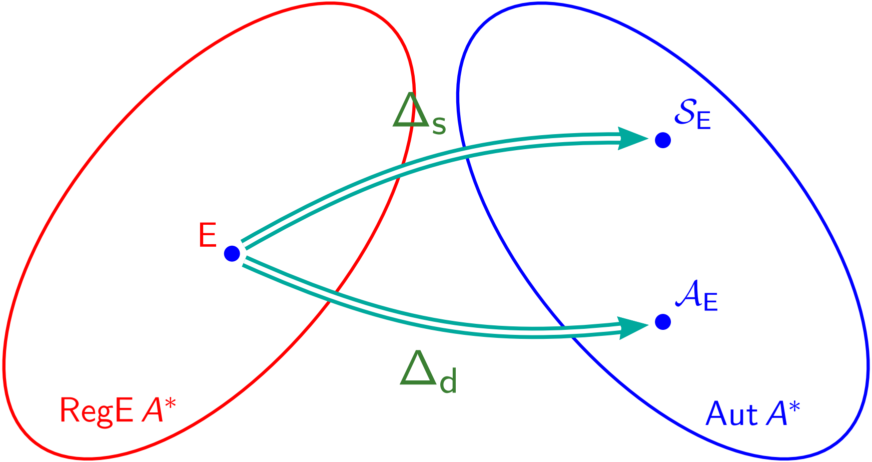
Theorem (B-K 92)

Computation of $\mathsf{E}^\bullet$ *is quadratic*

The derived term automaton of an expression

- Standard automaton of E — position, Glushkov
- Derived term automaton of E — Brzozowski–Antimirov



The Brzozowski derivatives

Preliminary

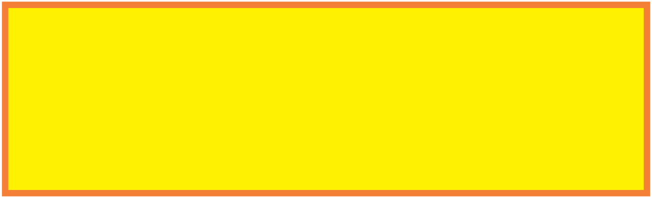
Construction of $\mathcal{A}_L$,

the *minimal (deterministic) automaton* of L

by means of the *quotients* of L

The Brzozowski derivatives

$\mathfrak{P}(A^*)$



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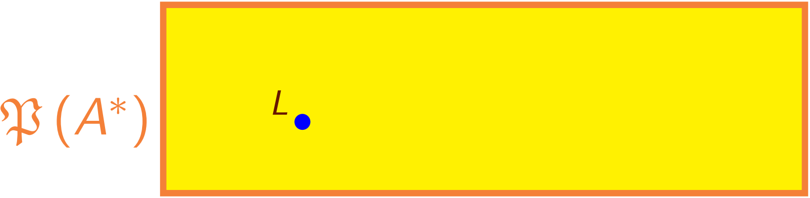

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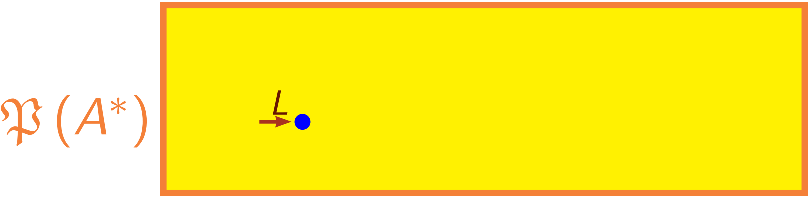

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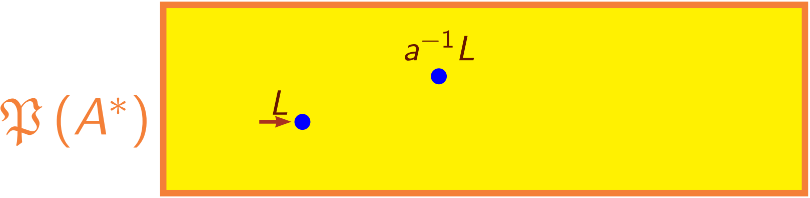

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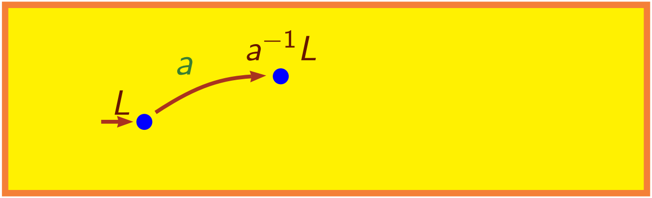

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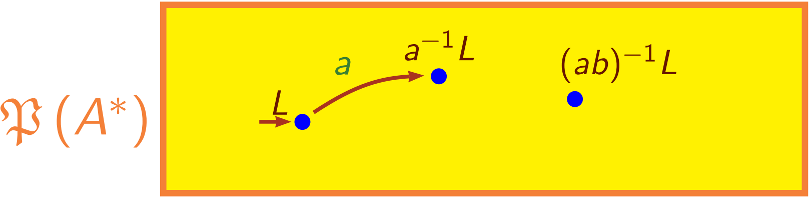

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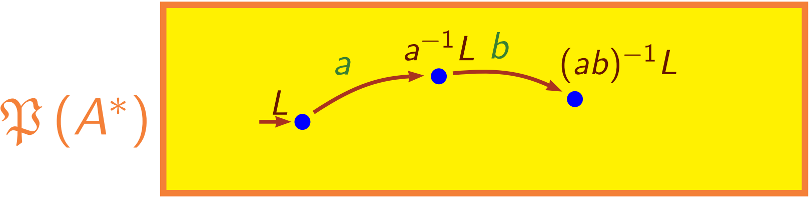

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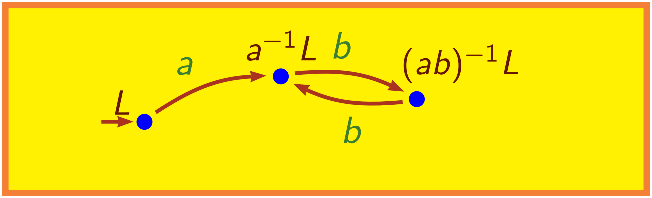

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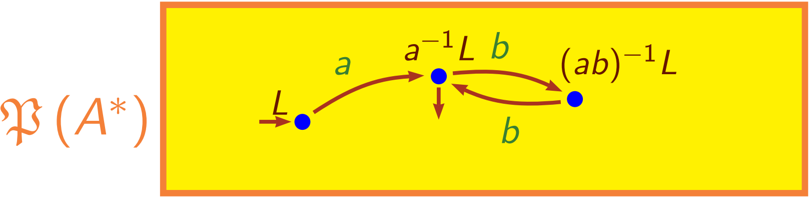

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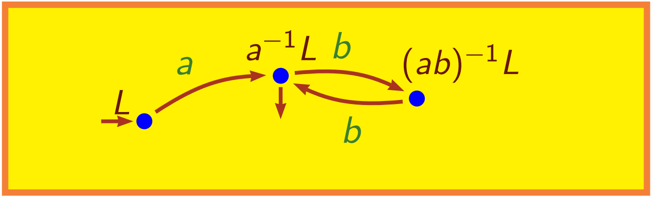

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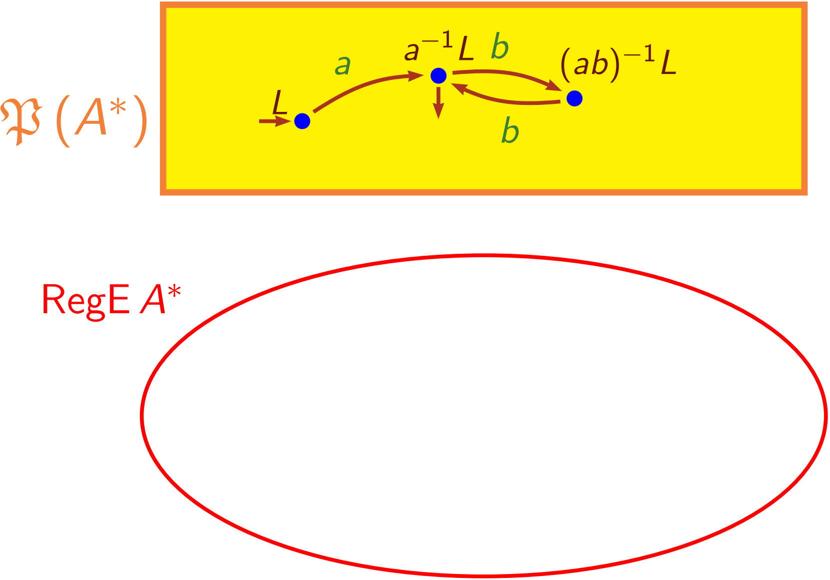

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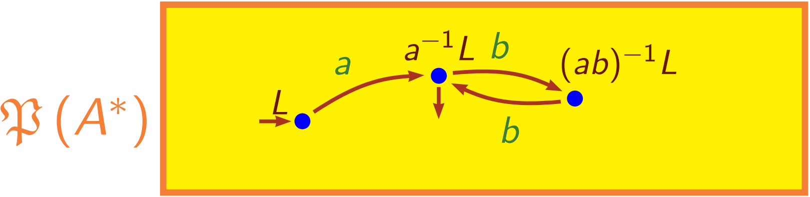

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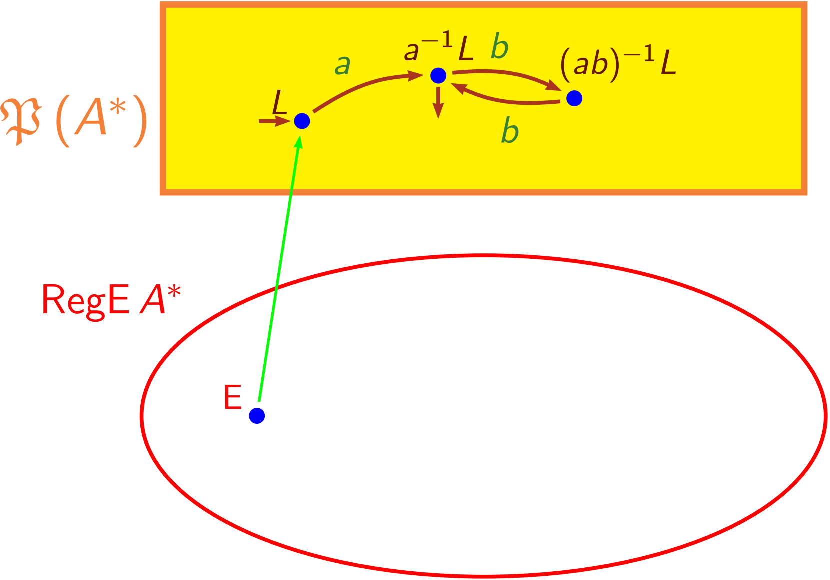

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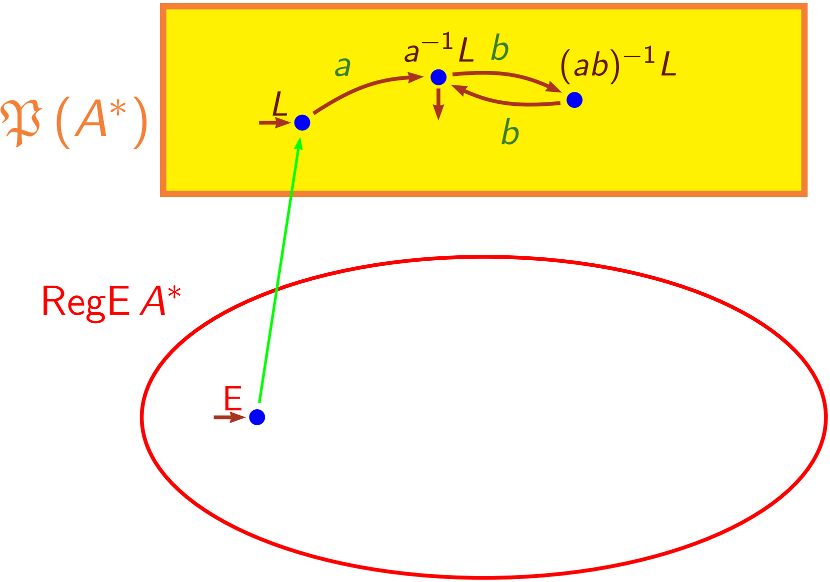

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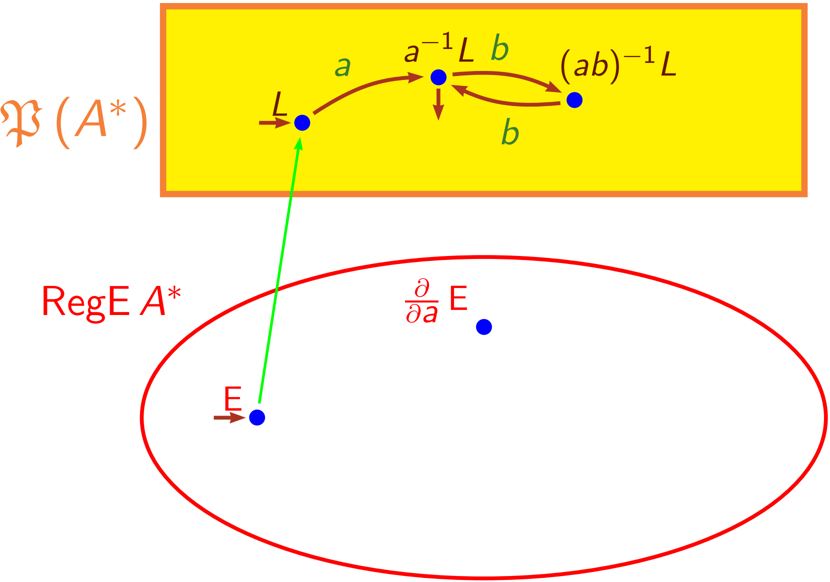

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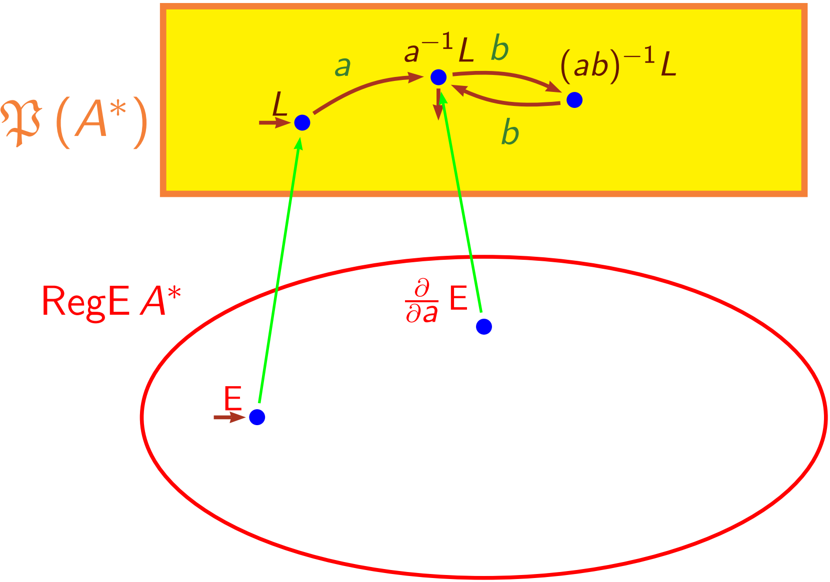

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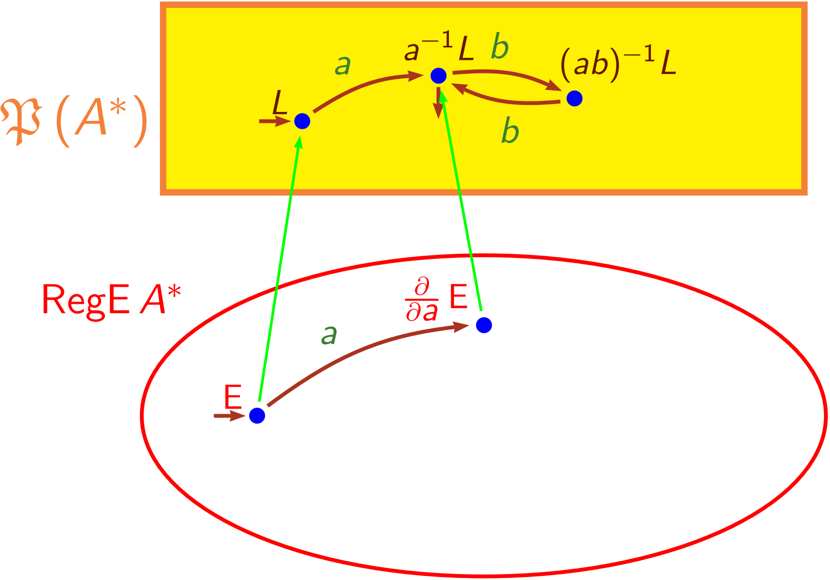

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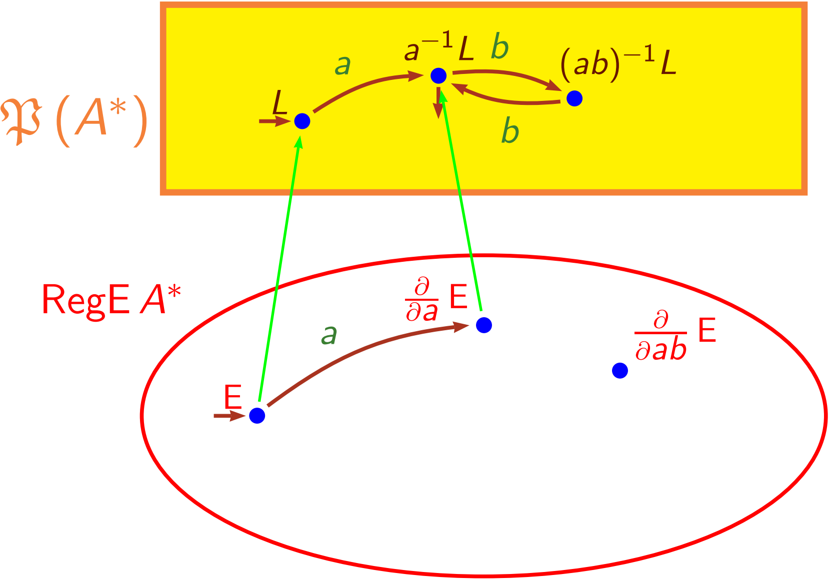

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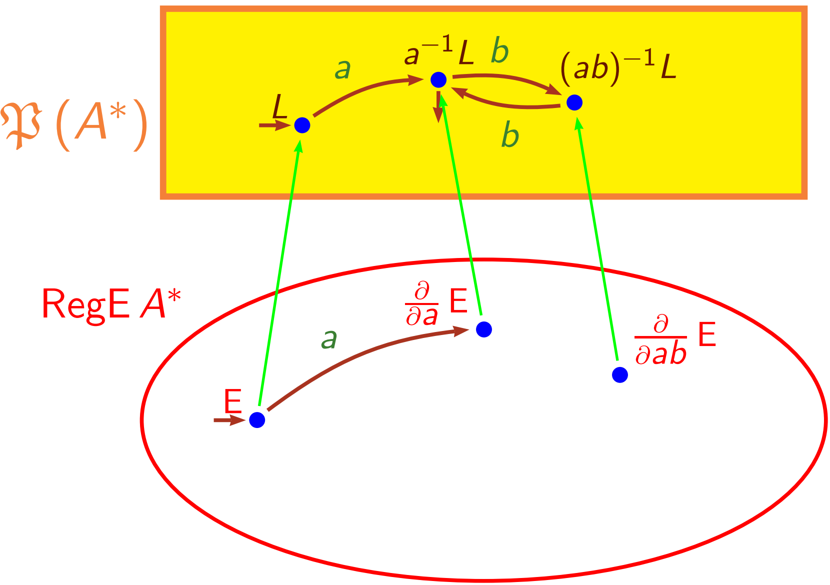

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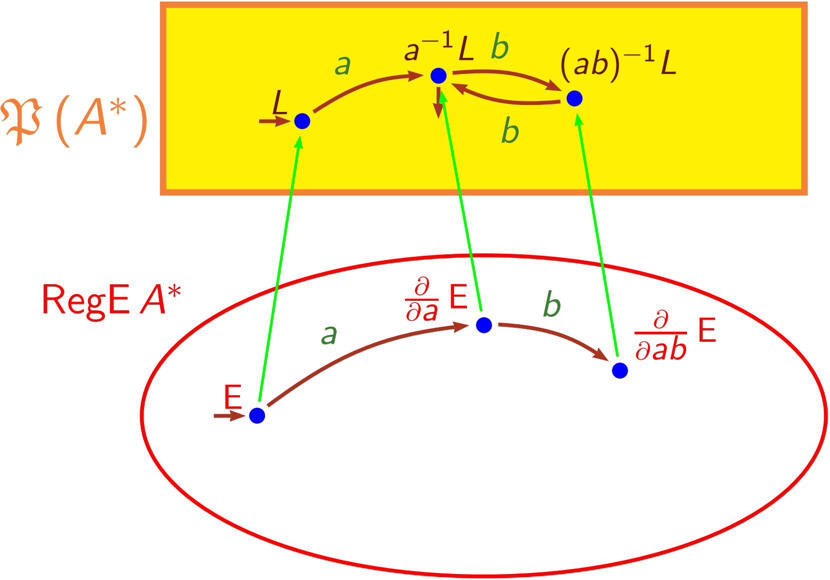

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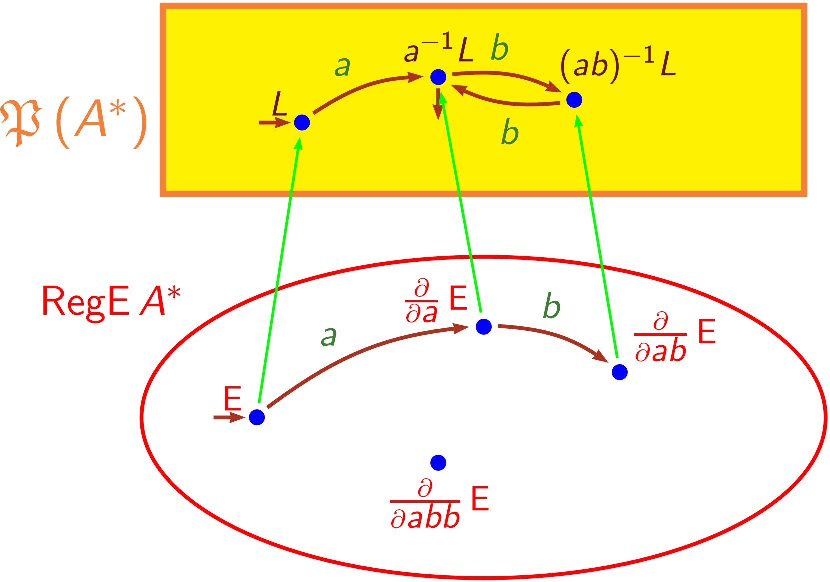

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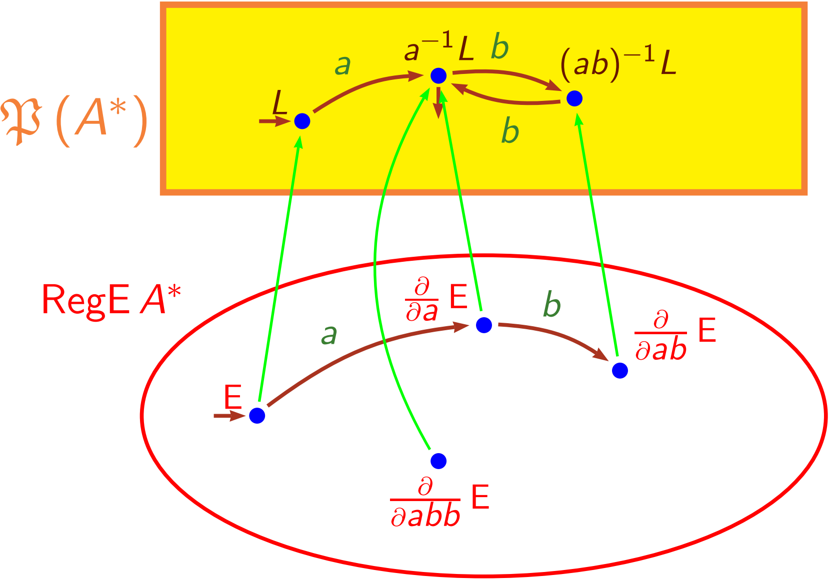

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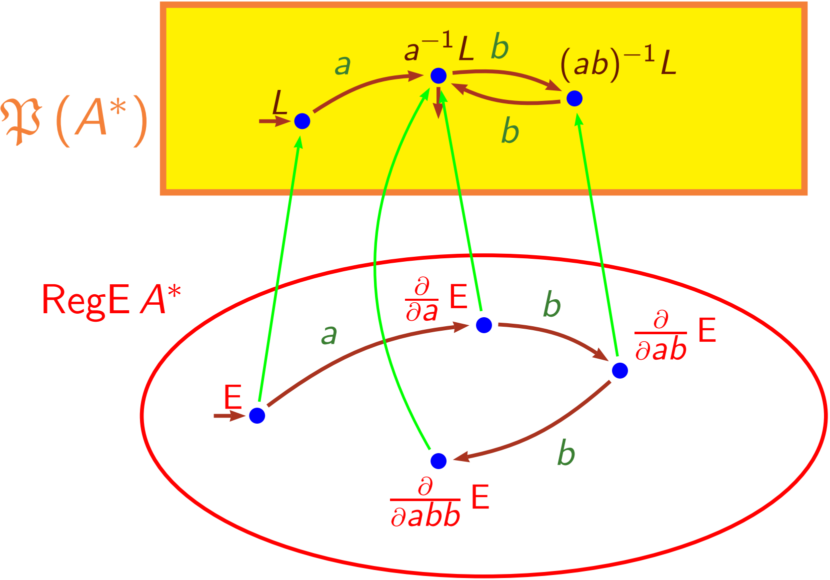

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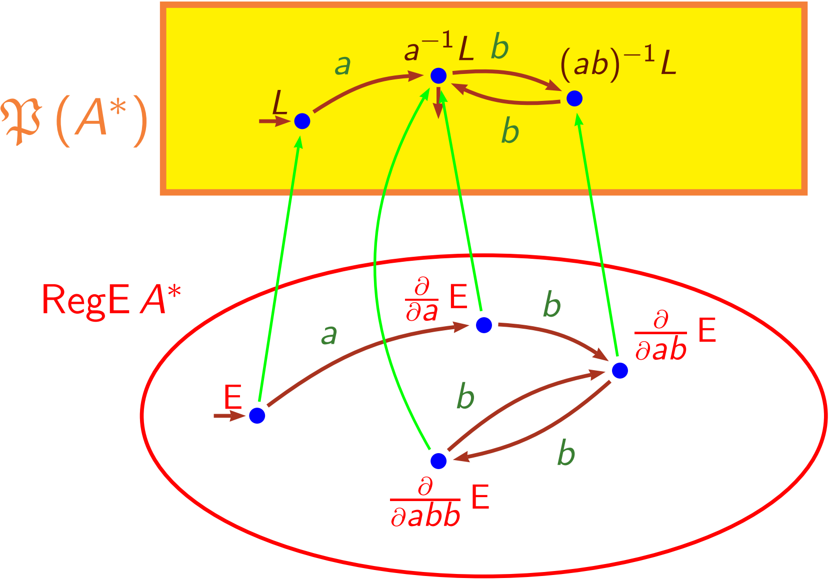

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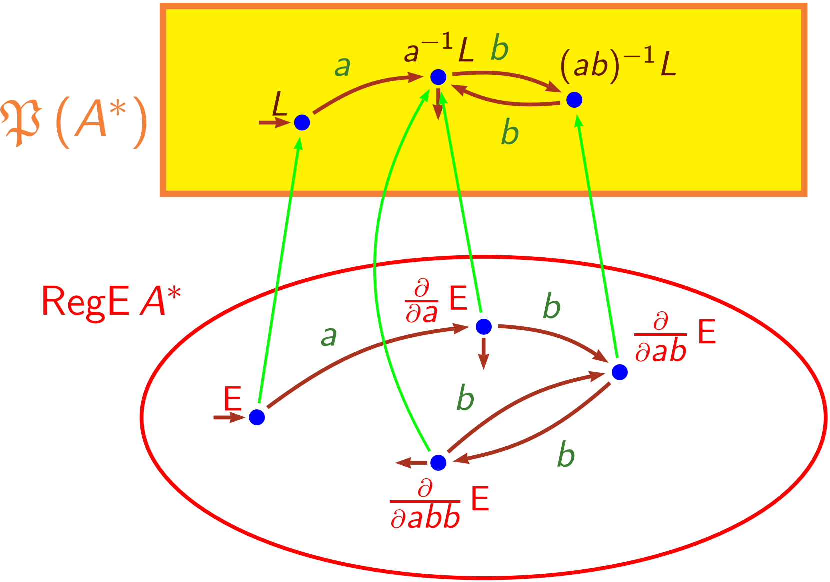

The Brzozowski derivation

Definition (Brzozowski 64)

$\mathsf{E} \in \mathsf{RegE}\, A^*$ $\quad \frac{\partial}{\partial a}\,\mathsf{E}$ is defined by induction.

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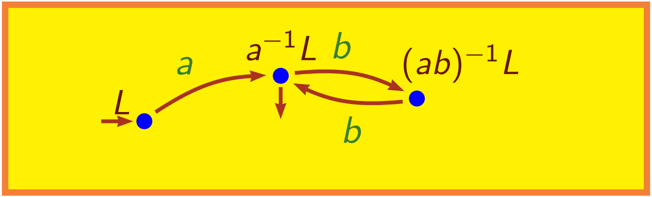
Theorem (Brzozowski 64)

For every E *, there is a finite number of derivatives modulo* **A** *,* **C** *, and* **I**

The Brzozowski–Antimirov derivation

$\mathfrak{P}(A^*)$



$\text{RegE}\, A^*$

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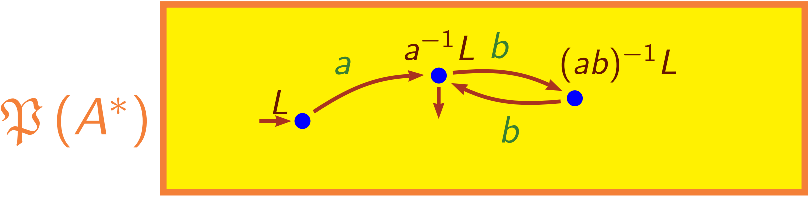

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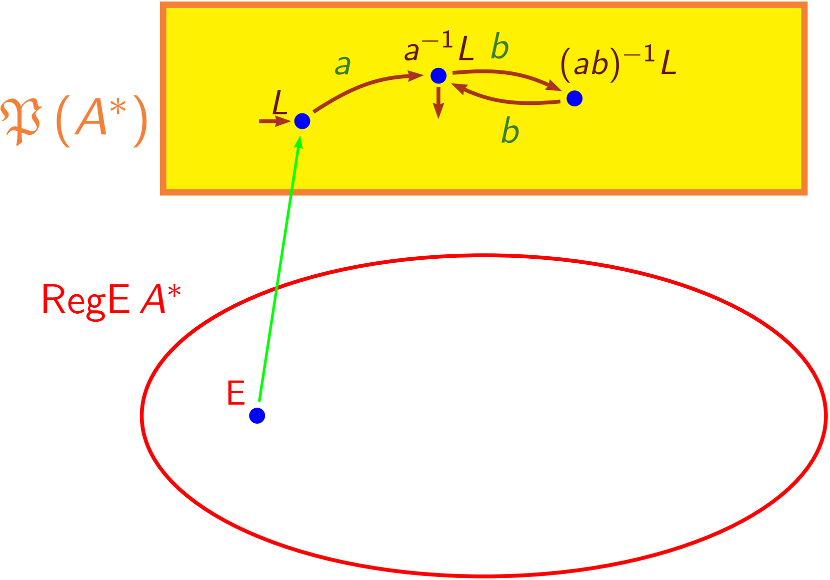

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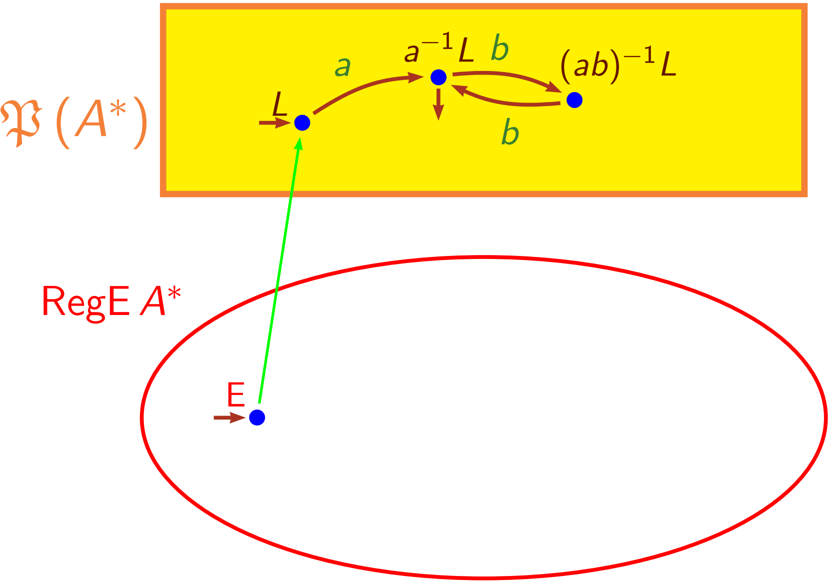

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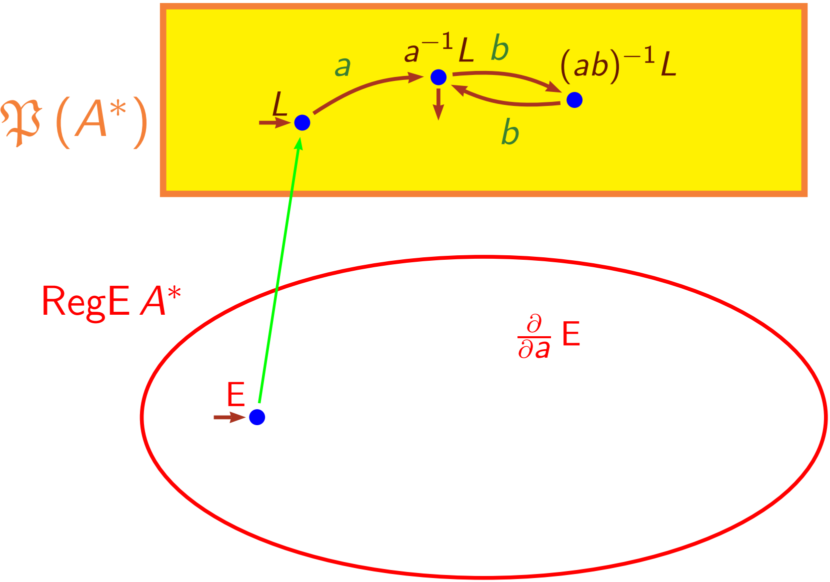

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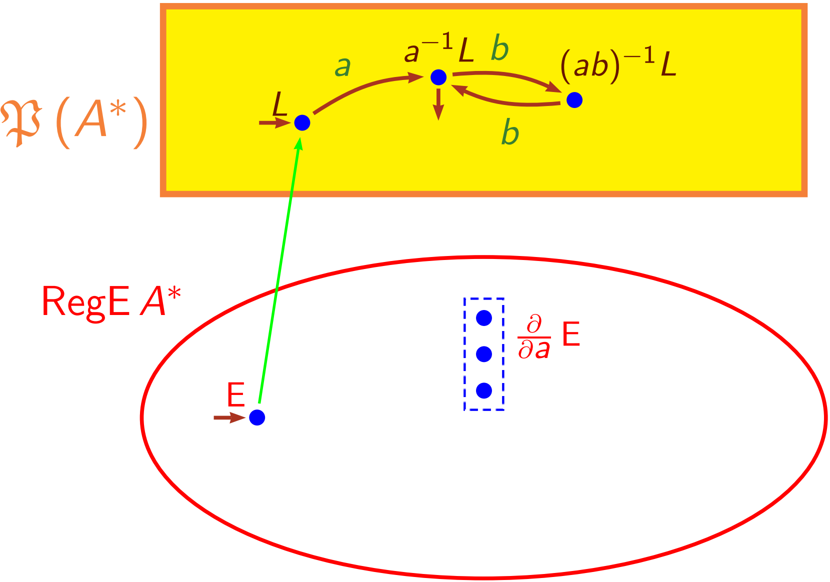

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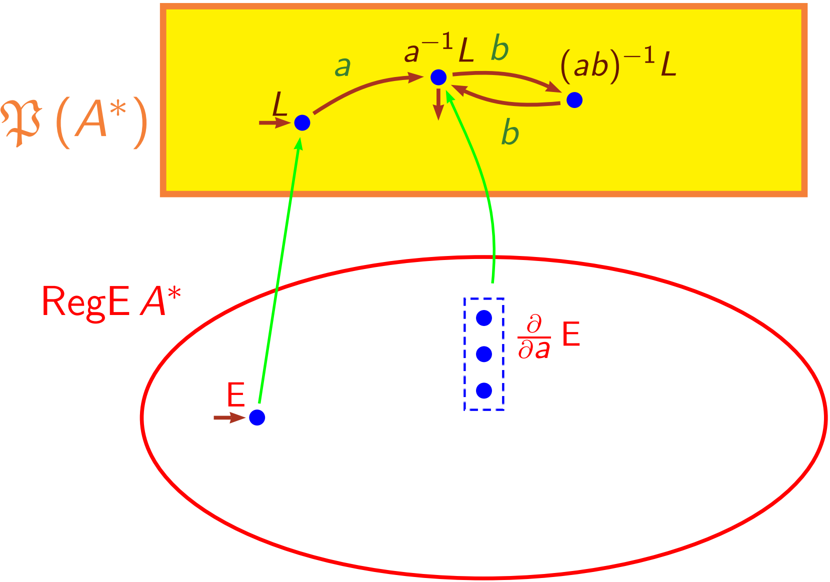

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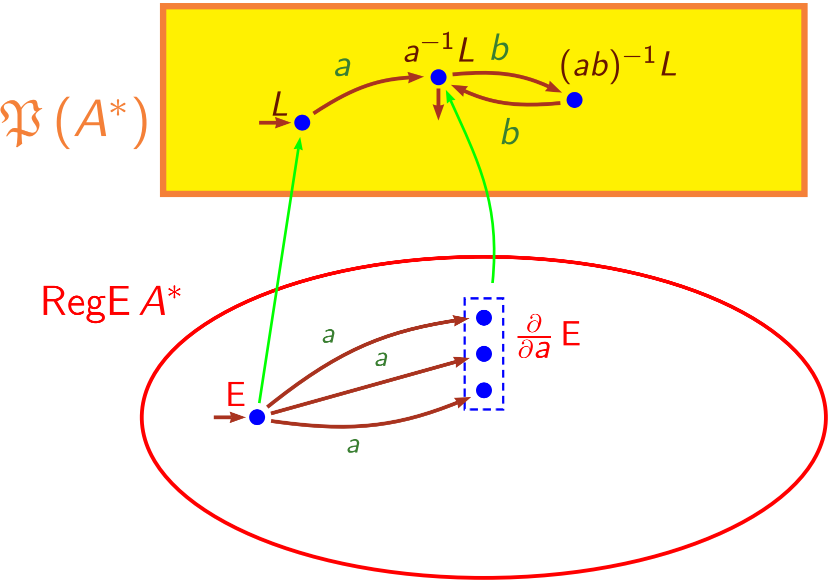

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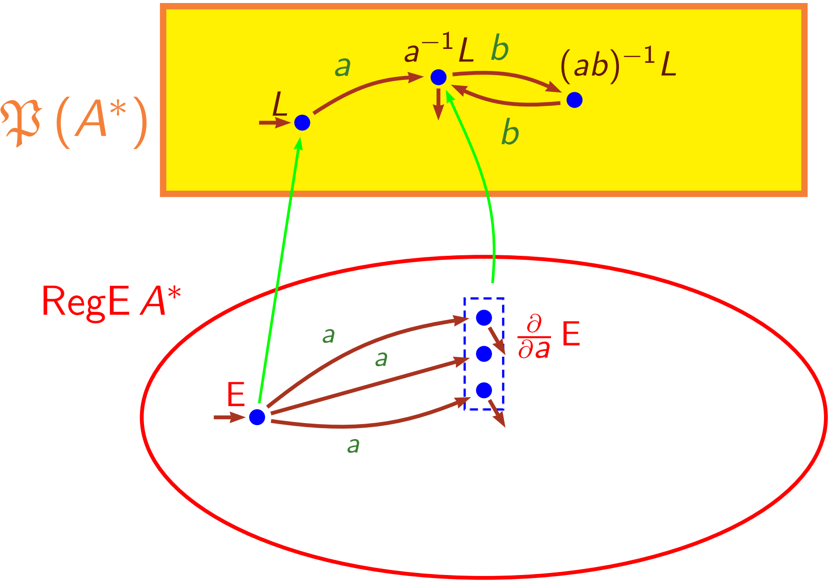

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$$\frac{\partial}{\partial a}(\mathsf{E}+\mathsf{F}) = \frac{\partial}{\partial a}\,\mathsf{E} \cup \frac{\partial}{\partial a}\,\mathsf{F}$$

$$\frac{\partial}{\partial a}(\mathsf{E}\cdot\mathsf{F}) = \left[\frac{\partial}{\partial a}\,\mathsf{E}\right]\cdot\mathsf{F} \cup \mathsf{c}(\mathsf{E})\,\frac{\partial}{\partial a}\,\mathsf{F}$$

$$\frac{\partial}{\partial a}(\mathsf{E}^*) = \left[\frac{\partial}{\partial a}\,\mathsf{E}\right]\cdot\mathsf{E}^*$$

$$\frac{\partial}{\partial a}\left[\bigcup_{i\in I}\mathsf{E}_i\right] = \bigcup_{i\in I}\frac{\partial}{\partial a}\,\mathsf{E}_i\,, \qquad \left[\bigcup_{i\in I}\mathsf{E}_i\right]\cdot\mathsf{F} = \bigcup_{i\in I}(\mathsf{E}_i\cdot\mathsf{F})\,.$$

$$\frac{\partial}{\partial fa}\,\mathsf{E} = \frac{\partial}{\partial a}\left(\frac{\partial}{\partial f}\,\mathsf{E}\right)$$

The Brzozowski–Antimirov derivation

Example $\mathsf{E}_1 = (a^* b + b\, b^* a)^*$

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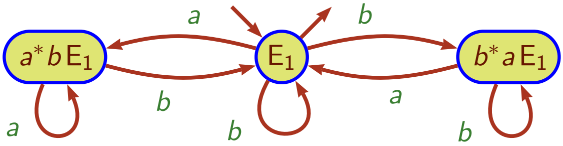

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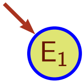

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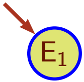

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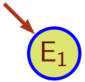

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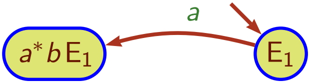

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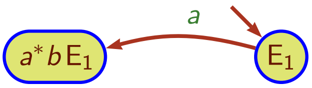

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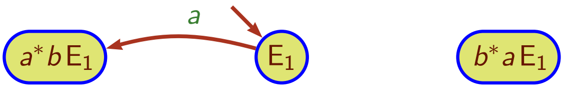

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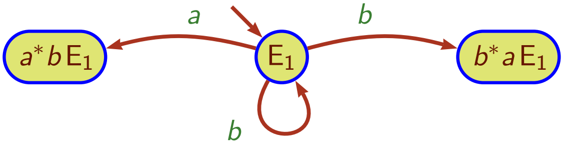

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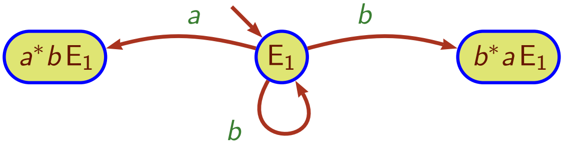

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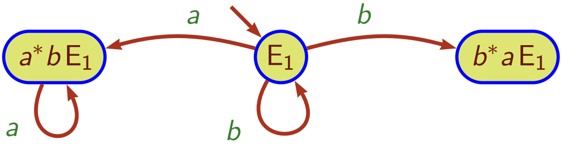

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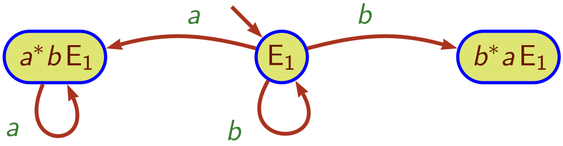

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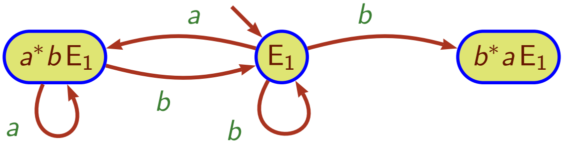

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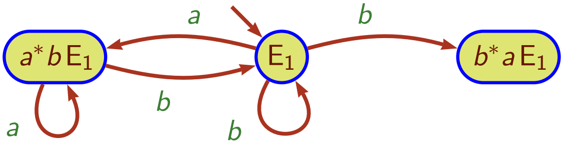

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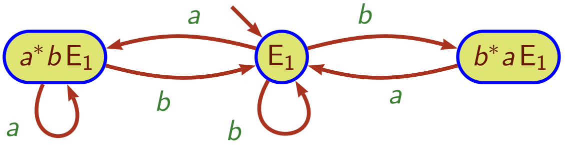

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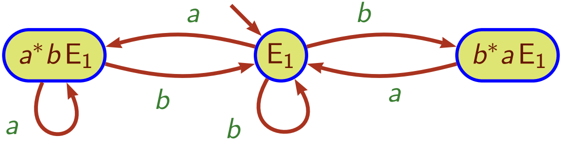

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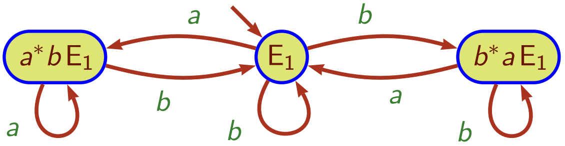

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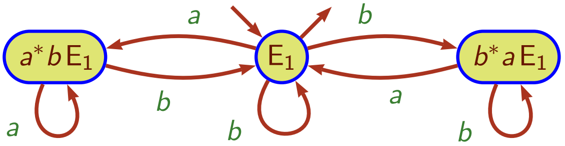

The Brzozowski–Antimirov derivation

Theorem (Antimirov 96)

$\mathcal{A}_{\mathsf{E}}$ *is an NFA which accepts* $L(\mathsf{E})$
and has less than $\ell(\mathsf{E}) + 1$ *states*

The Brzozowski–Antimirov derivation

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$\mathcal{B}_{\mathsf{E}}$ = Determinization of $\mathcal{A}_{\mathsf{E}}$

The Brzozowski–Antimirov derivation

Theorem (Antimirov 96)

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$\mathcal{B}_{\mathsf{E}}$ = Determinization of $\mathcal{A}_{\mathsf{E}}$ almost true

The Brzozowski–Antimirov derivation

Theorem (Antimirov 96)

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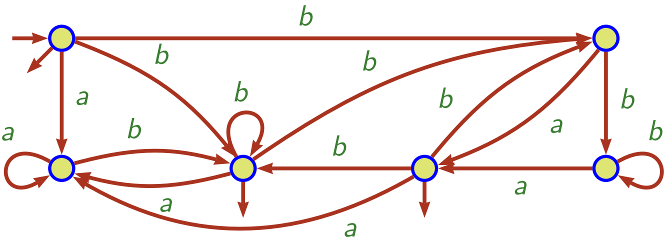
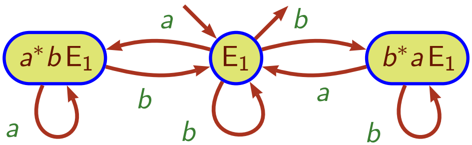
Theorem (Champarnaud–Ziadi 02)

$\mathcal{A}_{\mathsf{E}}$ *is a quotient of* $\mathcal{S}_{\mathsf{E}}$

The Brzozowski–Antimirov derivation





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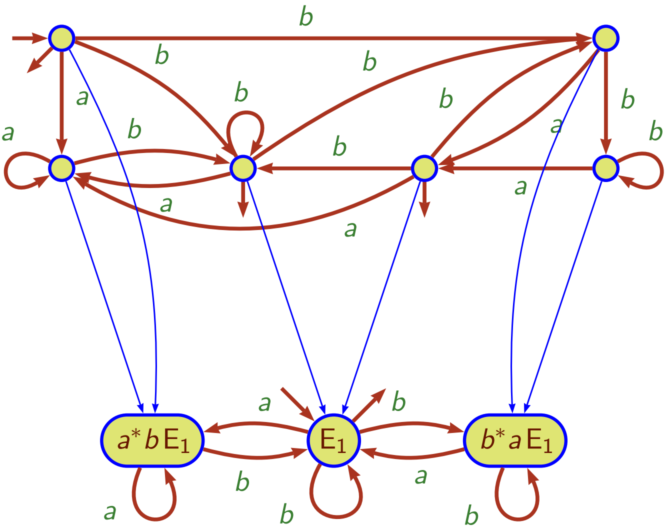

The Brzozowski–Antimirov derivation

Theorem (Champarnaud–Ziadi 01)

Computation of $\mathcal{A}_{\mathsf{E}}$ *is quadratic*

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Computation of $\mathcal{A}_{\mathsf{E}}$ *is quadratic*

Observation

Size of $\mathcal{A}_{\mathsf{E}}$ *much smaller than size of* $\mathcal{S}_{\mathsf{E}}$
when $\mathsf{E} = \Gamma(\mathcal{A})$ *, for a certain* $\mathcal{A}$

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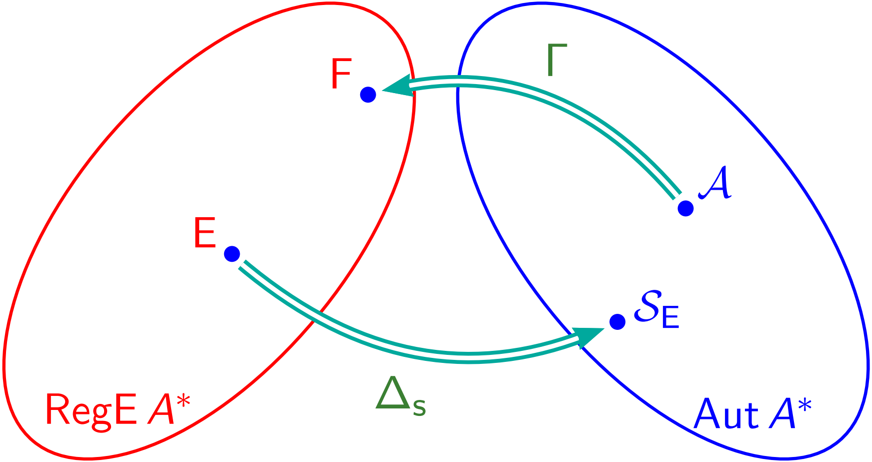
Observation

Even for $\mathsf{E} = \Gamma(\mathcal{A})$ *,*

computation of $\mathcal{S}_{\mathsf{E}}$ *followed by a quotient*

more effective than computation of $\mathcal{A}_{\mathsf{E}}$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Γ
$\mathcal{A}$
$\mathcal{S}_\mathsf{E}$
Δ_s
RegE A^*
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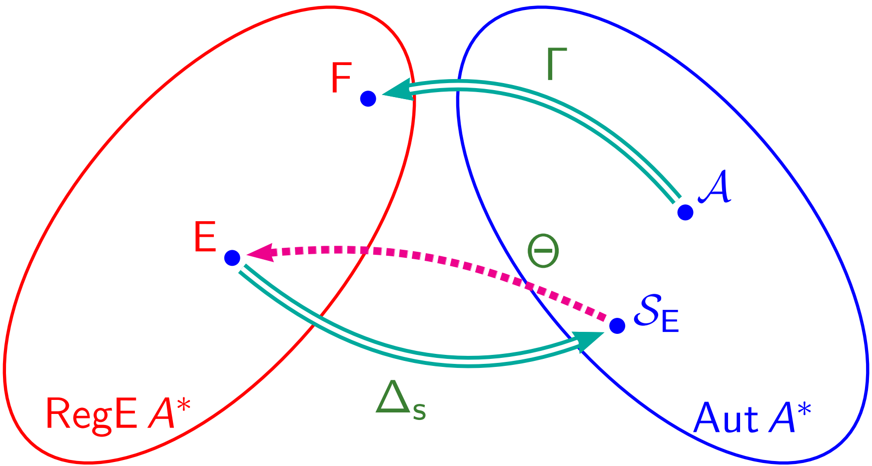

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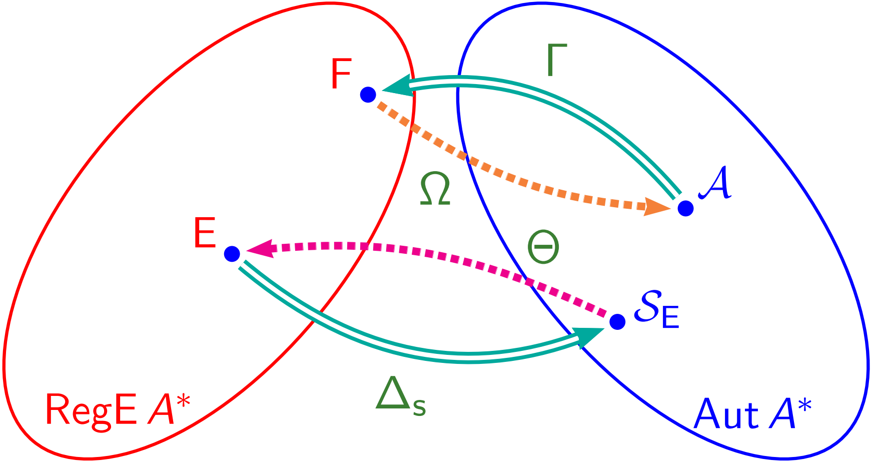

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