We start by recalling and discussing the following result, which can be found in [Bourbaki TG, chap. III, exercices, §6, (4)] as well as [Warner 1993, corollary 10.16 on p. 79]:

Theorem 1. Let A be a Hausdorff topological ring and $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ two summable families of elements of A: then *if* the family $(x_i\, y_j)_{(i,j)\in I\times J}$ is also summable, we have

$$\Big(\sum_{i\in I} x_i\Big)\Big(\sum_{j\in J} y_j\Big) = \sum_{(i,j)\in I\times J} x_i\, y_j \tag{$*$}$$

In the above statement, a *topological ring* A refers to a ring that is also a topological space in such a way that the ring operations $(x,y)\mapsto x+y$ and $x\mapsto -x$ and $(x,y)\mapsto xy$ are continuous (from A^2 or A to A). A family $(z_k)_{k\in K}$ of elements of A is then said to be *summable* with sum z when the following holds: for every neighborhood U of z, there exists a finite $K_0\subseteq K$ such that for every finite $K_0\subseteq K_1\subseteq K$ the sum $\sum_{k\in K_1} z_k$ belongs to U; equivalently, in more sophisticated terms, the net taking a finite subset $K_1\subseteq K$ to the sum $\sum_{k\in K_1} z_k$ converges to z (the net being indexed by the directed set of finite subsets K_1 of K). When A is separated, the sum z is evidently unique, so we can denote this by $\sum_{k\in K} z_k = z$.

We record the following easy proof of the above theorem, which, in fact, shows that the statement still holds when A is merely assumed to be a topological semiring (a *semiring* is essentially a ring without subtraction: namely, an commutative monoid written additively together with a monoid written multiplicatively, such that multiplication distributes over addition and that the unit for addition 0 is multiplicatively absorbing; a topological semiring is a semiring that is also a topological space in such a way that operations $(x,y)\mapsto x+y$ and $(x,y)\mapsto xy$ are continuous):

Proof of theorem 1. If I_1 and J_1 are finite subsets of I and J respectively, then finite distributivity gives $\big(\sum_{i\in I_1} x_i\big)\big(\sum_{j\in J_1} y_j\big) = \sum_{(i,j)\in I_1\times J_1} x_i\, y_j$. Now continuity of the product at (x,y) where $x := \sum_{i\in I} x_i$ and $y := \sum_{j\in J} y_j$ shows that, because $I_1\mapsto \sum_{i\in I_1} x_i$ tends to x and $J_1\mapsto\sum_{j\in J_1} y_j$ tends to y, so also $(I_1,J_1)\mapsto \big(\sum_{i\in I_1} x_i\big)\big(\sum_{j\in J_1} y_j\big)$ tends to xy, in other words $(I_1,J_1)\mapsto \sum_{(i,j)\in I_1\times J_1} x_i\, y_j$ tends to xy (where, again, I_1 and J_1 range over the finite subsets of I and J respectively). What we want to show is that $K_1\mapsto\sum_{(i,j)\in K_1} x_i\, y_j$ tends to xy where K_1 ranges over the finite subsets of $I\times J$. Now the hypothesis made ensures that $K_1\mapsto \sum_{(i,j)\in K_1} x_i\, y_j$ has some limit $z\in A$; and the finite rectangles $I_1\times J_1$ are cofinal in the final subsets of $I\times J$ (in the sense that for every finite $K_0\subseteq I\times J$ there exists $I_0\subseteq I$ and $J_0\subseteq J$ finite such that $K_0\subseteq I_0\times J_0$), so clearly $(I_1,J_1)\mapsto\sum_{(i,j)\in I_1\times J_1} x_i\, y_j$ tends to z: by uniqueness of the limit, $z = xy$ and we are done. ☺

As it is written, the statement of theorem 1 raises the question of whether assuming the summability of $(x_i\, y_j)_{(i,j)\in I\times J}$ is indeed necessary. One might believe that it is not, on account of the following classical fact:

Proposition 2. Let $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ be two summable families of real numbers: then the family $(x_i\, y_j)_{(i,j)\in I\times J}$ is also summable (so $(*)$ holds in theorem 1).

This in turn depends crucially on the following:

Proposition 3. A family $(z_k)_{k\in K}$ of real numbers is summable (in the sense recalled above) if and only if it is absolutely summable in the sense that $\sum_{k\in K}|z_k| < +\infty$ (where $\sum_{k\in K}|z_k|$, which is always meaningful as an element of $[0,+\infty]$, is the least upper bound of the finite sums $\sum_{k\in K_1}|z_k|$ for $K_1\subseteq K$ finite).

Proof. If $\sum_{k\in K} |z_k| < +\infty$ then $K_1 \mapsto \sum_{k\in K_1} |z_k|$ is Cauchy (in the sense that for every $\varepsilon > 0$ there exists $K_0 \subseteq K$ finite such that whenever K_1, K_2 are finite subsets of K containing K_0, the sums $\sum_{k\in K_1} |z_k|$ and $\sum_{k\in K_2} |z_k|$ are ε-close). By the triangle inequality, this implies that $K_1 \mapsto \sum_{k\in K_1} z_k$ is also Cauchy, and by completeness, it converges, i.e., $(z_k)_{k\in K}$ is summable.

Conversely, if $(z_k)_{k\in K}$ is summable, then by letting K_+ be the set $\{k \in K : z_k \geq 0\}$ of indices of nonnegative terms and K_- the set $\{k \in K : z_k < 0\}$ of negative ones, the nets $K_1 \mapsto \sum_{k\in K_1} z_k$ are Cauchy for K_1 ranging over the finite subsets of K_+ or K_- respectively, so they are convergent, meaning that $(z_k)_{k\in K_+}$ and $(z_k)_{k\in K_-}$ are both convergent, so $(z_k)_{k\in K_+}$ and $(-z_k)_{k\in K_-}$ are, which means that $(|z_k|)_{k\in K}$ is convergent, so of course $\sum_{k\in K} |z_k| < +\infty$. ☺

We note that the proof of the "if" part ("absolutely summable implies summable") remains valid in any Banach space, whereas the "only if" part depends crucially on the ordered structure of the real numbers: it remains true in $\mathbb{R}^n$ (reasoning componentwise), but, as we shall recall below, does not even hold in separable Hilbert spaces.

Proof of proposition 2. Assuming $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ are summable, we have $\sum_{i\in I} |x_i| < +\infty$ and $\sum_{j\in J} |y_j| < +\infty$ by the "only if" part of proposition 3, so $\sum_{i\in I} \Big(|x_i|\big(\sum_{j\in J} |y_j|\big)\Big) < +\infty$ (multiplying the first family by the sum of the second), so $\sum_{i\in I} \big(\sum_{j\in J} |x_i|\,|y_j|\big) < +\infty$, so $\sum_{(i,j)\in I\times J} |x_i|\,|y_j| < +\infty$ by Tonelli's theorem on summable families of nonnegative real numbers, in other words $\sum_{(i,j)\in I\times J} |x_i\, y_j| < +\infty$, and $(x_i\, y_j)_{(i,j)\in I\times J}$ is summable by the "if" part of proposition 3 ☺

We now proceed to give an example showing that the hypothesis of summablity of $(x_i\, y_j)_{(i,j)\in I\times J}$ is indeed necessary in theorem 1, even if A is a commutative unital Banach algebra. The crucial construction is given by the following:

Proposition 4. There exists a continuous symmetric bilinear form $\varphi\colon E \times E \to \mathbb{R}$ (even with $|\varphi(z, z')| \leq \|z\|\,\|z'\|$), where $E = \ell^2$ is the (real) Hilbert space of square-summable sequences of real numbers, and a family $(z_k)_{k\in\mathbb{N}}$ with values in E which is summable but such that $(\varphi(z_i, z_j))_{(i,j)\in\mathbb{N}^2}$ is not summable.

We will prove the proposition below. Let us first show how it is used to construct the desired counterexample:

Proposition 5. There exists a commutative unital Banach algebra A, and a family $(z_k)_{k\in\mathbb{N}}$ with values in A which is summable but such that $(z_i\, z_j)_{(i,j)\in\mathbb{N}^2}$ is not summable.

Proof. As a real vector space, we let $A = \mathbb{R} \oplus E \oplus \mathbb{R}'$, where $E = \ell^2$ and $\mathbb{R}'$ refers to another copy of $\mathbb{R}$: we write elements (t, z, u) of A simply as $t + z + u$, where it is understood that $t \in \mathbb{R}$ and $z \in E$ and $u \in \mathbb{R}'$. The norm of such an element will simply be $|t| + \|z\| + |u|$. The unit element of A is the element $1 \in \mathbb{R}$ (meaning $(1, 0, 0)$). Multiplication is defined by letting $(t + z + u)\,(t' + z' + u') = tt' + (tz' + t'z) + (tu' + t'u + \varphi(z, z'))$ where φ is given by proposition 4 (considered to have values in $\mathbb{R}'$, as well as tu' and $t'u$). Every property of being a Banach algebra is obvious except perhaps associativity; but because of distributivity (=bilinearity), we need only check the latter for the various cases where each of the three factors are assuned to be in $\mathbb{R}$, E or $\mathbb{R}'$: if any one is in $\mathbb{R}$ then associativity follows from the fact that $1 \in \mathbb{R}$ is the unit element, and in any other case the product of three factors is zero so associativity is trivial. Commutativity of A follows from symmetry of φ.

Taking the family (z_k) given by proposition 4 (considered as elements of E inside A), it is obvious that they are still summable as elements of A, and that their products $z_i\, z_j$, which are given by $\varphi(z_i, z_j)$ (considered as elements of $\mathbb{R}'$ inside A), are not summable. ☺

So we now turn to the proof of proposition 4, which we take, with only minor modifications, from [Ryan 2002, example 4.30].

We start by recalling the following easy fact, which is also classical:

Lemma 6. Let E be a Hilbert space. Then a family $(z_k)_{k\in K}$ of *mutually orthogonal* vectors ($z_k \perp z_{k'}$ whenver $k \neq k'$) in E is summable iff $\sum_{k\in K} \|z_k\|^2 < +\infty$.

Proof. If (z_k) is summable then $K_1 \mapsto \sum_{k\in K_1} z_k$ has a limit (where K_1 ranges over the finite subsets of K), so $K_1 \mapsto \big\| \sum_{k\in K_1} z_k \big\|^2$ has one, and by the Pythagorean theorem this is $K_1 \mapsto \sum_{k\in K_1} \|z_k\|^2$, so $\sum_{k\in K} \|z_k\|^2 < +\infty$.

Conversely, if $\sum_{k\in K} \|z_k\|^2 < +\infty$, the net $K_1 \mapsto \sum_{k\in K_1} \|z_k\|^2$ is convergent, so it is Cauchy, and another application of the Pythagorean theorem shows that $K_1 \mapsto \sum_{k\in K_1} z_k$ is Cauchy, so it converges, meaning that (z_k) is summable. ☺

Now we proceed with the construction itself.

Proof of proposition 4. Let $E = \ell^2$, and let (e_i) be its standard Hilbert basis. For convenience, we will take i to range over the set $\mathbb{N}_{>0}$ of nonzero natural numbers.

For $r \in \mathbb{N}$, we define I_r to be the interval of natural numbers between 2^r and $2^{r+1} - 1$ inclusive (so $I_0 = \{1\}$ and $I_1 = \{2, 3\}$ and $I_2 = \{4, 5, 6, 7\}$ and so on): it has cardinality $\#I_r = 2^r$. When $i \in \mathbb{N}_{>0}$, we let $r(i)$ be the unique r such that $i \in I_r$ (in other words, the exponent of the largest power of 2 which is $\leq i$).

We define $z_i := \frac{1}{2^{r(i)/2}\,(1+r(i))}\, e_i$. So the z_i are mutually orthogonal, and $\sum_{i\in I_r} \|z_i\|^2 = \frac{1}{(1+r)^2}$, which by lemma 6 shows that $(z_i)_{i\in\mathbb{N}_{>0}}$ is summable in ℓ^2 (the sum being, of course, the sequence z with values $\frac{1}{2^{r(i)/2}\,(1+r(i))}$, whose square norm is $\pi^2/6$ but this is unimportant).

To define the form φ, we first consider the $2^r \times 2^r$ real matrices M_r with values in $\{\pm 1\}$ ("Sylvester's Hadamard matrices" or "Walsh matrices") defined inductively by $M_0 = (1)$ (the unit 1×1 matrix) and

$$M_{r+1} := \begin{pmatrix} M_r & M_r \\ M_r & -M_r \end{pmatrix}$$

The rows (or columns) of M_r are mutually orthogonal and each one has Euclidean norm $2^{r/2}$, meaning that the matrix $2^{-r/2}\, M_r$ is an orthogonal matrix; also, M_r is symmetric. For the sake of notational convenience, we will consider the 2^r rows and columns of M_r to be indexed by I_r (so we write $(M_r)_{i,j}$ for the element in row i and column j of M_r, when $i, j \in I_r$).

We define $\varphi(u, v)$ as the bilinear form given by the block diagonal $2^{-r/2}\, M_r$, meaning that, writing $(u_i)_{i\in\mathbb{N}_{>0}}$ and $(v_i)_{i\in\mathbb{N}_{>0}}$ for the coefficients of u and v respectively,

$$\varphi\Big(\sum_{i=1}^{+\infty} u_i e_i,\ \sum_{i=1}^{+\infty} v_i e_i\Big) := \sum_{r=0}^{+\infty} \Big(2^{-r/2} \sum_{(i,j)\in(I_r)^2} (M_r)_{i,j}\, u_i\, v_j\Big) \qquad (\dagger)$$

We need to check that the right-hand side of (†) makes sense (converges absolutely) and is in absolute value at most $\|u\|\,\|v\|$ whenever $u, v \in \ell^2$: but since $2^{-r/2}\, M_r$ is orthogonal, the Cauchy-Schwarz inequality shows that $2^{-r/2} \sum_{i,j\in I_r} (M_r)_{i,j}\, u_i\, v_j$ is in absolute value at most equal to $\|u\|_{I_r}\, \|v\|_{I_r}$ where

$\|w\|_{I_r} := \sum_{i\in I_r} w_i^2$ stands for the Euclidean norm of the I_r-indexed components of u; and again by the Cauchy-Schwarz inequality (applied to the square-summable sequences $\|u\|_{I_r}$ and $\|v\|_{I_r}$ of reals), $\sum_{r=0}^{+\infty} \|u\|_{I_r} \|v\|_{I_r} \leq \big(\sum_{r=0}^{+\infty} \|u\|_{I_r}^2\big)^{1/2} \big(\sum_{r=0}^{+\infty} \|v\|_{I_r}^2\big)^{1/2} = \|u\| \, \|v\|$. So φ is indeed well-defined and is a continuous bilinear form with $|\varphi(u, v)| \leq \|u\| \, \|v\|$. Symmetry of φ follows from symmetry of the M_r.

Lastly, we need to check that $\varphi(z_i, z_j)$ is not summable. By proposition 3, it is enough to see that $\sum_{(i,j)\in(\mathbb{N}_{>0})^2} |\varphi(z_i, z_j)|$ diverges. Now $\varphi(z_i, z_j)$ is $\frac{(M_r)_{i,j}}{2^{3r/2}\,(1+r)^2}$ when i, j belong to the same I_r (there are two factors $2^{-r/2}$ coming from each of the z_k and a third one coming from the definition of φ) and zero otherwise. Since $(M_r)_{i,j} \in \{\pm 1\}$, the absolute value of $\varphi(z_i, z_j)$ is therefore $\frac{1}{2^{3r/2}\,(1+r)^2}$: summing over the 2^{2r} pairs $(i, j) \in (I_r)^2$, we get $\frac{2^{r/2}}{(1+r)^2}$. This certainly diverges, so $\sum_{(i,j)\in(\mathbb{N}_{>0})^2} |\varphi(z_i, z_j)|$ diverges. ☺

(We mention in passing that, with the notations of the above proof, $\gamma := \varphi(z, z) = \varphi\big(\sum_{i=1}^{+\infty} z_i,\ \sum_{j=1}^{+\infty} z_j\big)$ can be computed from (†) by noting that $\sum_{(i,j)\in(I_r)^2} (M_r)_{i,j} = 2^r$: so we get $\gamma = \sum_{r=0}^{+\infty} \frac{1}{2^{r/2}\,(1+r)^2}$, a fairly uninteresting real number with value $\sqrt{2}\ \mathrm{Li}_2(\frac{1}{\sqrt{2}}) \approx 1.28$, but which is indeed less than $\|z\|^2 = \pi^2/6 \approx 1.64$. The crucial difference between computing γ, which consists of summing the $\varphi(z_i, z_j)$ *in a certain way*, and trying to sum the $|\varphi(z_i, z_j)|$, is that in one case we use $\sum_{(i,j)\in(I_r)^2} (M_r)_{i,j} = 2^r$ whereas in the other we stumble upon $\sum_{(i,j)\in(I_r)^2} |(M_r)_{i,j}| = 2^{2r}$. We can sum the $\varphi(z_i, z_j)$ provided we sum all values for $(i, j) \in (I_r)^2$ together.)

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