Segmentation using Deformable Models

Isabelle Bloch

LIP6, Sorbonne Université - On leave from LTCI, Télécom Paris

isabelle.bloch@sorbonne-universite.fr

Introduction

Deformable models = evolution of curves / surfaces to minimize an energy function for

- finding the best image partition into homogeneous regions
- finding the contours of an object

Representation:

- parametric,
- implicit, using level sets.

Criteria on

- contours
- region homogeneity
- other types of constraints:
  - regularity (internal)
  - external: balloon force, spatial relations, geometry...

Parametric active contours

- First work: Kass, Witkins et Terzopoulos (1987).
- Principle: evolution of a curve under internal and external forces (one object, parametric representation)

$$v(s) = [x(s), y(s)]^t \quad s \in [0, 1]$$

- Energy: $E_{total} = \int_0^1 (E_{int}(v(s)) + E_{image}(v(s)) + E_{ext}(v(s)))\, ds$
- Internal energy: regularization

$$E_{int} = \alpha(s) \left(\frac{dv}{ds}\right)^2 + \beta(s) \left(\frac{d^2 v}{ds^2}\right)^2$$

control of tension (length of the curve) and of curvature (if $s =$ curvilinear coordinate, tangent $T = \frac{dv}{ds}$, $||T|| = 1$, and $\frac{dT}{ds} = kN$, k curvature).
- Energy from image information (gradient, information on contours):

$$E_{image} = g(||\nabla f||)$$

- External energy: many possibilities

Resolution

- Euler-Lagrange equation:

$$\frac{\partial E}{\partial v} - \frac{d}{ds}(\frac{\partial E}{dv'}) + \frac{d^2}{ds^2}(\frac{\partial E}{\partial v''}) = 0$$

$\Rightarrow$ variational problem

$$-(\alpha v')'(s) + (\beta v'')''(s) + \nabla P(v) = 0$$

$$P(v) = E_{image}(v) + E_{ext}(v) \qquad F(v) = -\nabla P(v)$$

$+$ limit conditions.

- Discretization using finite differences

$$V^t = [v_0^t,\ v_1^t,\ v_2^t,\ \ldots\ldots,\ v_{n-1}^t]^t$$

$$\frac{\beta}{h^2} v_{i+2} - (\frac{\alpha}{h} + 4\frac{\beta}{h^2}) v_{i+1} + (\frac{2\alpha}{h} + 4\frac{6\beta}{h^2}) v_i - (\frac{\alpha}{h} + 4\frac{\beta}{h^2}) v_{i-1} + \frac{\beta}{h^2} v_{i-2} = F(v_i)$$

$$AV = F$$

A: pentadiagonal matrix

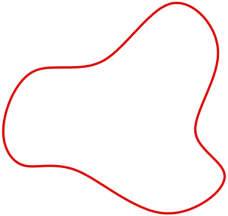
Resolution (dynamic scheme with inertia)

- Evolution of the curve in time:

$$\tau \frac{\partial v}{\partial t} = -\alpha v'' + \beta v'''' + F(v)$$

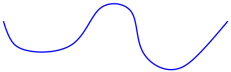
$$v(t+1) = (A + \tau I)^{-1}(F(v(t)) + \tau v(t))$$

- initialization (crucial for convergence)
- choice of τ (inertia, regularization of A)
- matrix inversion
- constant discretization step for t (set to have a displacement of 1-2 pixels)
- stopping criterion



Closed active contour

$$
\begin{bmatrix}
2\alpha+6\beta & -\alpha-4\beta & \beta & 0 & 0 & \beta. & -\alpha-4\beta \\
-\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & 0 & \ldots & \\
\beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & & \\
0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & & \\
0 & 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & & \\
\ldots & \ldots & \ldots & & & \ldots & \\
 & & & & & & \\
-\alpha-4\beta & \beta & 0 & 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta
\end{bmatrix}
$$

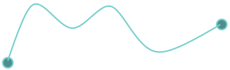


Free end-points

$$
\begin{bmatrix}
\beta & -2\beta & \beta & 0 & 0 & 0 & 0 \\
-\alpha-2\beta & 2\alpha+5\beta & -\alpha-4\beta & \beta & 0 & \cdots & \\
\beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & & \\
0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & & \\
0 & 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & & \\
\cdots & \cdots & \cdots & & & \cdots & \\
 & & & \beta & -\alpha-4\beta & 2\alpha+5\beta & -\alpha-2\beta \\
0 & 0 & 0 & & \beta & -2\beta & \beta
\end{bmatrix}
$$

(with $v''(0) = v''(1) = v'''(0) = v'''(1) = 0$)

Fixed end-points



$$
\begin{bmatrix}
2\alpha+6\beta & -\alpha-4\beta & \beta & 0 & 0 & .0 & .0 \\
-\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & 0 & \dots & \\
\beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & & \\
0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & & \\
0 & 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & & \\
\dots & \dots & \dots & & & \dots &
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & 0 & 0 & & \beta & -2\ \beta & \overset{\dots}{\beta}
\end{bmatrix}
$$

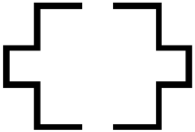
Example

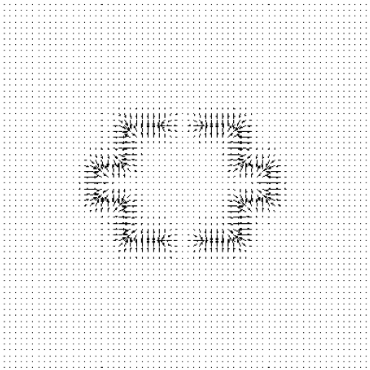
Evolution example

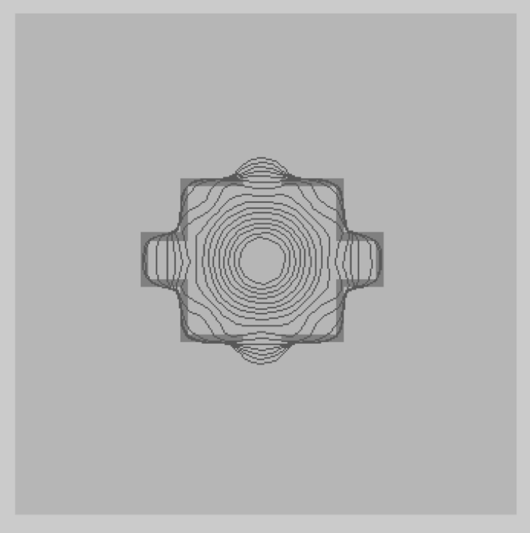
Evolution

Balloon force (Cohen, 1991)

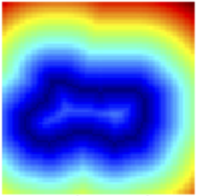
- Problem:
  - bad initialization $\Rightarrow$ no attraction
  - no forces $\Rightarrow$ curve collapses
- Solution using a pression force (balloon): $k_1 N(s)$
  ($N(s)$ unit normal vector at s).
- Initialization: inside or outside the object (not necessarily very close to the searched object)







Constraint on distance to some edges



$D(x,y)$

Distance map $D(x,y) \Rightarrow$ potential

$$P_{dist}(x,y) = we^{-D(x,y)}$$

$$F_{ext} = -\nabla P_{dist}$$

Gradient vector flow (Xu et Prince, 1997)

- Objectives:
  - get rid of the constraints on the initialization
  - convergence towards concave regions
- Diffusion of gradients in the whole image
- For $\overrightarrow{v}(x, y) = (u(x, y), v(x, y))$, minimization of:

$$E = \int\int \mu(u_x^2 + u_y^2 + v_x^2 + v_y^2) + |\nabla f(x, y)|^2 |\overrightarrow{v} - \nabla f(x, y)|^2 dxdy$$

- f = contour map

Solving and generalizing GVF

$$\mu\nabla^2 u - (u - f_x)(f_x^2 + f_y^2) = 0$$
$$\mu\nabla^2 v - (v - f_y)(f_x^2 + f_y^2) = 0$$

More general formulation:

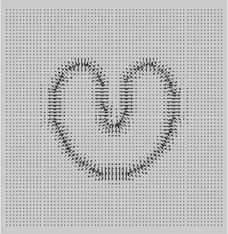
$$\frac{\partial v}{\partial t} = g(\|\nabla f\|)\nabla^2 v - h(\|\nabla f\|)(v - \nabla f)$$

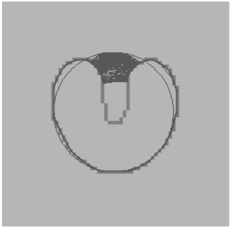
$$v(x, y, 0) = \nabla f(x, y)$$

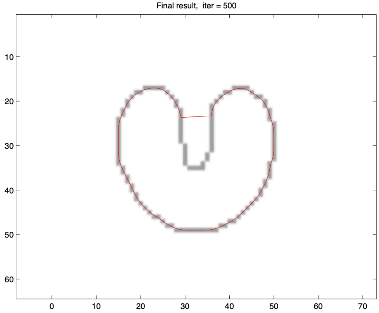
Examples for functions g and h:

- $g(r) = \mu$, $h(r) = r^2$
- $g(r) = \exp(\frac{-r^2}{k})$, $h(r) = 1 - g(r)$

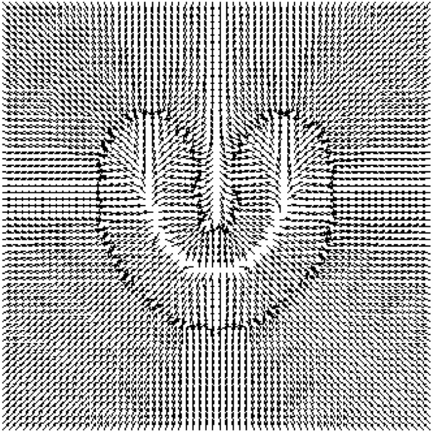
Classical active contour: illustration

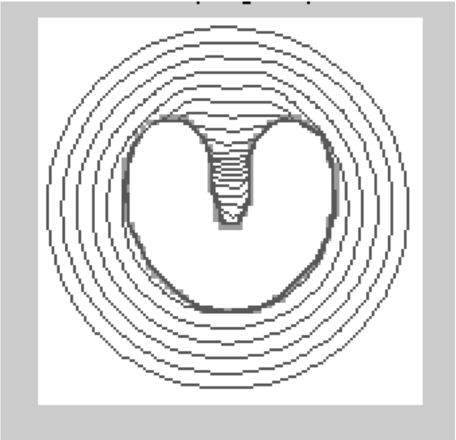




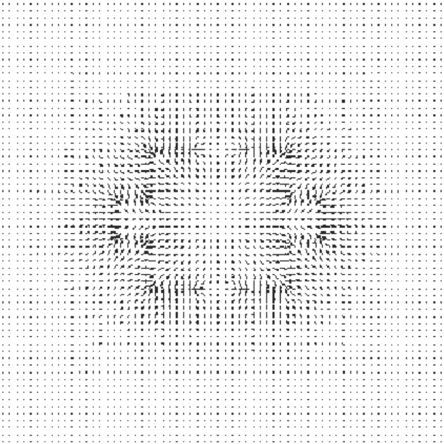


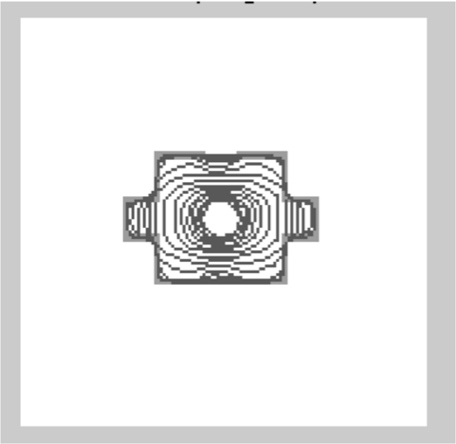
Using GVF:





Using GVF:

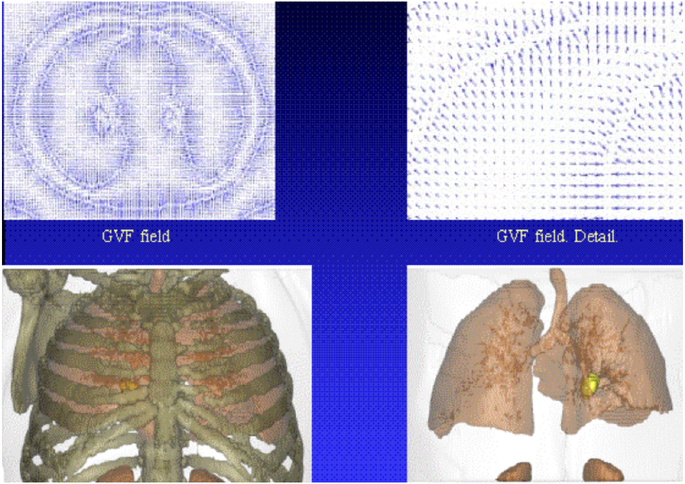


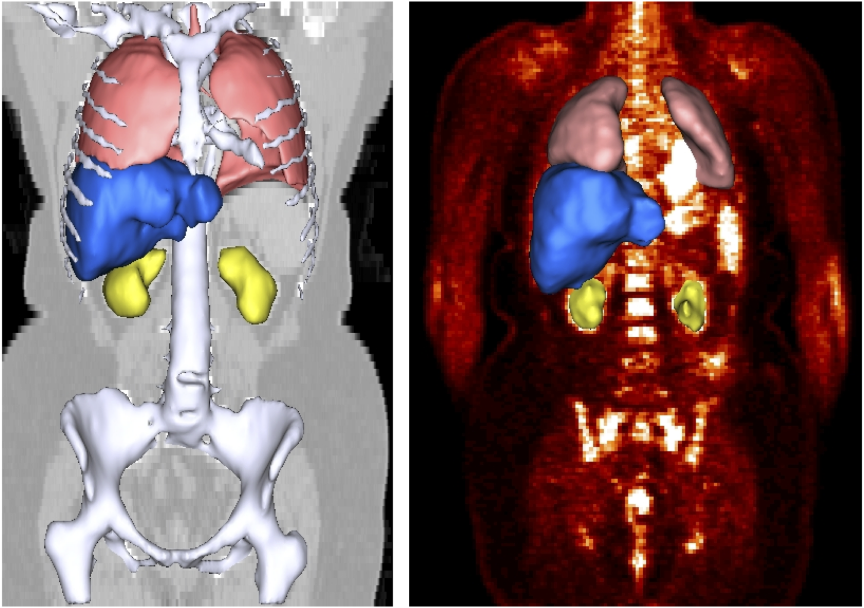


GVF: example of evolution

Evolution

GVF: example in medical imaging





3D parametric deformable models

- Segmentation with regularization:
  - data fidelity
  - smooth surface
- Minimization of energy:

$$\begin{aligned} E(v) &= \int_\Omega w_{10} \| \frac{\partial v}{\partial r} \|^2 + w_{01} \| \frac{\partial v}{\partial s} \|^2 \\ &\quad + w_{20} \| \frac{\partial^2 v}{\partial r^2} \|^2 + w_{02} \| \frac{\partial^2 v}{\partial s^2} \|^2 + 2 w_{11} \| \frac{\partial^2 v}{\partial r \partial s} \|^2 drds \\ &\quad + \int_\Omega P(v) drds \end{aligned}$$

  - first order: elastic membrane (curvature)
  - second order: thin plate (torsion)
  - P: attraction potential

Similar resolution schemes as in 2D.

Geodesic active contours (Caselles, 1997)

- Principle:

$$J_1(v) = \alpha \int_a^b |v'(s)|^2 ds + \lambda \int_a^b g(|\nabla I(v(s))|)^2 ds$$

$$\text{minimization of } J_2(v) = 2\sqrt{\lambda\alpha} \int_a^b |v'(s)| g(|\nabla I(v(s))|) ds$$

$\Rightarrow$ computation of geodesics according to a new metric (induced by the image)

- Evolution equation:

$$\frac{\partial v}{\partial t} = g(I)\kappa N - (\nabla g.N)N$$

Geodesic active contours: example

Evolution

Level sets

Implicit representation, without parametrization

- Principle:
  Let $\Gamma(t)$ be a closed hypersurface (dimension $d-1$)
  Let ψ (dimension d) be a function taking values in $\mathbb{R}$ with

$$\Gamma(t) = \{x \in \mathbb{R}^d \mid \psi(x,t) = 0\}$$

  propagation of Γ (evolution along the normal) $\Leftrightarrow$ propagation of ψ
- Example: distance function
- Evolution equation of ψ

$$\frac{\partial \psi}{\partial t} = -F||\nabla \psi||$$

NB: normal $N = \frac{\nabla \psi}{||\nabla \psi||}$, mean curvature $k = div(\frac{\nabla \psi}{||\nabla \psi||})$

Additional dimension: example of distance

Propagation using level sets

- Propagation speed:

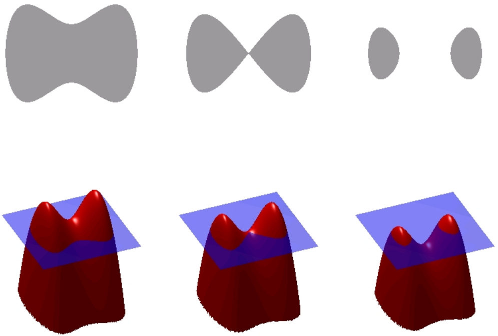
$$F = (F_A + F_G)k_I$$

  - F_A expansion or contraction, independent of geometry,
  - F_G geometric properties (curvature),
  - k_I stopping criterion (image).

$$F = \frac{1}{1 + ||\nabla G_\sigma * I||^p}(\pm 1 + \epsilon\kappa)$$

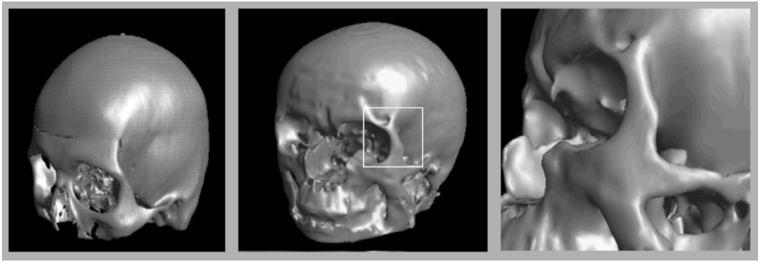
- Advantage: potential modification of topology.
- Speeding-up by computation only in a narrow band.

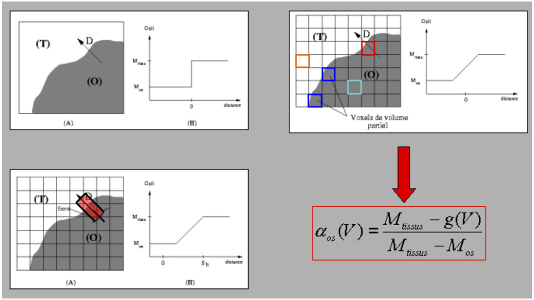
Level sets and change of topology



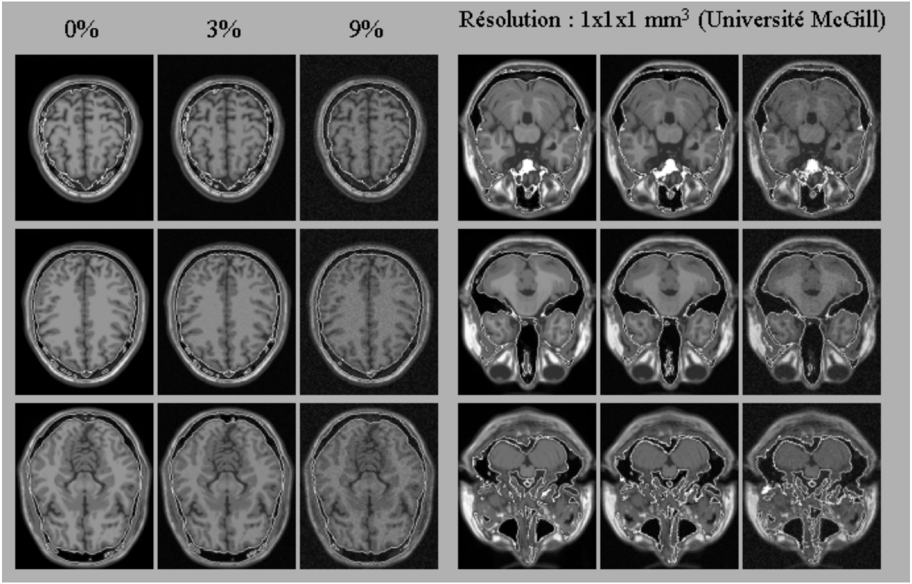
Example

Example: bone segmentation in MRI (H. Rifai)

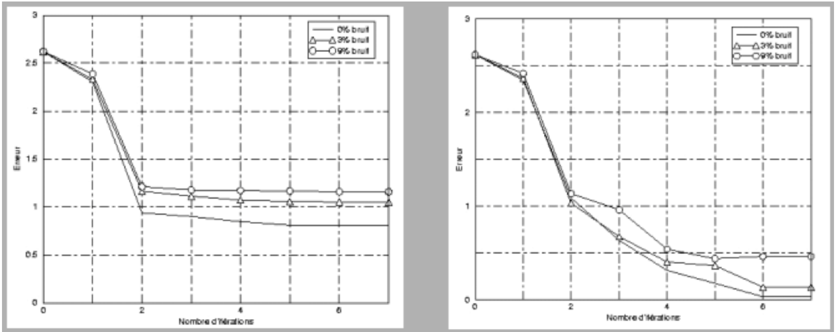




Simulated images:



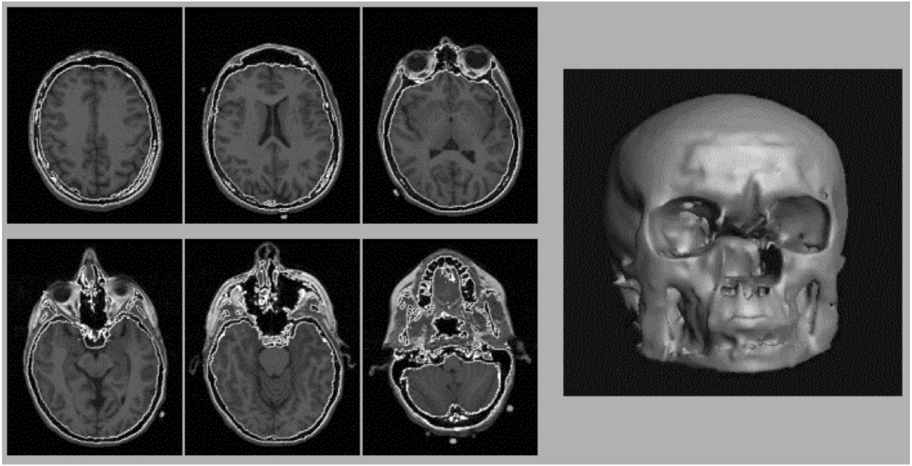
Segmentation error on simulated images with various noise levels:



Sans estimation du volume partiel.

Avec estimation du volume partiel.

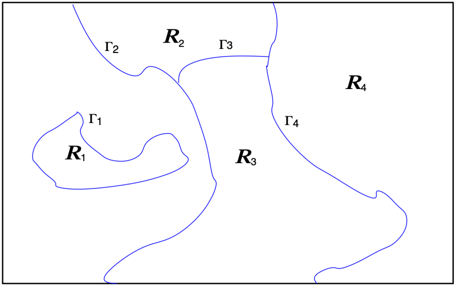
Real images:



Evolution

Result

- Image f on a support $\mathcal{I}$.
- Approximation by smooth functions $g(x, y)$ on regions R_i limited by contours Γ_j ($\mathcal{I} = \cup_i R_i \cup \Gamma$, $\Gamma = \cup_j \Gamma_j$):



- Functional to be minimized: $U(\Gamma, g, f) =$

$$\lambda \iint_{\mathcal{I}\setminus\Gamma} (f(x,y) - g(x,y))^2 dxdy + \mu \iint_{\mathcal{I}\setminus\Gamma} \|\nabla g(x,y)\|^2 dxdy + \nu \int_\Gamma dl$$

Mumford et Shah approach in the piecewise constant case

$g_i = constant$ on each R_i :

$$U_0(\Gamma, f) = \sum_i \lambda_i \iint_{R_i} (f - g_i)^2 dxdy + \nu \int_\Gamma dl$$

$$\Rightarrow g_i = \frac{1}{s_i} \iint_{R_i} f(x, y) dxdy$$

where s_i is the area of R_i

$\Rightarrow$ Partition of space into homogeneous regions, characterized by their average gray level.

Formulation using level sets

- Contour Γ
- Two regions $R_1 = int(\Gamma)$ and $R_2 = ext(\Gamma)$ with constant values g_1 and g_2

$$U(\Gamma, g, f) = \lambda_1 \iint_{R_1} (f - g_1)^2 dxdy + \lambda_2 \iint_{R_2} (f - g_2)^2 dxdy + \nu \int_\Gamma dl$$

Level set:

$$\phi(x,y) \begin{cases} = 0 & \text{on } \Gamma \\ > 0 & \text{in } R_1 = int(\Gamma) \\ < 0 & \text{in } R_2 = ext(\Gamma) \end{cases}$$

$$\Gamma(t) = \{\phi(t) = 0\}$$

Evolution (mean curvature motion):

$$\frac{\partial \phi}{\partial t} = |\nabla \phi| \nabla (\frac{\nabla \phi}{|\nabla \phi|}) \quad \phi(0) = \phi_0$$

$(v(r,s) = (r, s, \Phi(r,s)) \Rightarrow \frac{\partial v}{\partial r} = (1, 0, \frac{\partial \Phi}{\partial r})^t, \frac{\partial v}{\partial s} = (0, 1, \frac{\partial \Phi}{\partial s})^t$, etc.)

$$H(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases} \qquad H'(z) = \delta(z)$$

$$\Rightarrow \int_\Gamma dl = \int_\mathcal{I} |\nabla H(\phi)| dxdy = \int_\mathcal{I} \delta(\phi) |\nabla \phi| dxdy$$

and $U(\Gamma, g, f) =$

$$\lambda_1 \iint_\mathcal{I} (f - g_1)^2 H(\phi) dxdy + \lambda_2 \iint_\mathcal{I} (f - g_2)^2 (1 - H(\phi)) dxdy + \nu \int_\mathcal{I} \delta(\phi) |\nabla \phi| dxdy$$

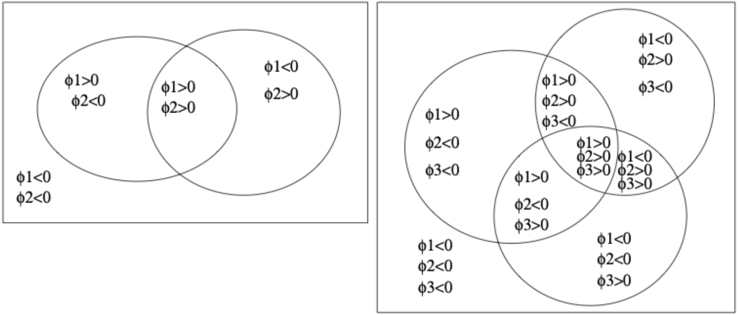
Minimization of U:

$$g_1 = \frac{\int_\mathcal{I} f H(\phi) dxdy}{\int_\mathcal{I} H(\phi) dxdy} \qquad g_2 = \frac{\int_\mathcal{I} f (1 - H(\phi)) dxdy}{\int_\mathcal{I} (1 - H(\phi)) dxdy}$$

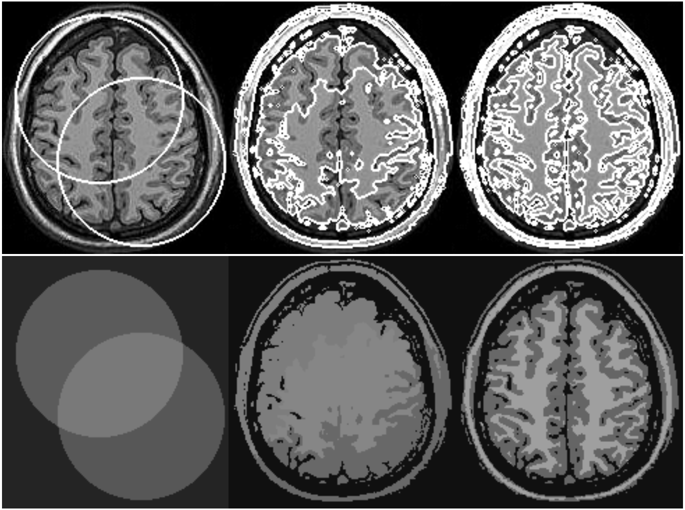
$$\frac{\partial \phi}{\partial t} = \delta(\varphi) [\nu \nabla (\frac{\nabla \phi}{|\nabla \phi|}) - \lambda_1 (f - g_1)^2 + \lambda_2 (f - g_2)^2]$$

In practice: smooth versions of δ and H.

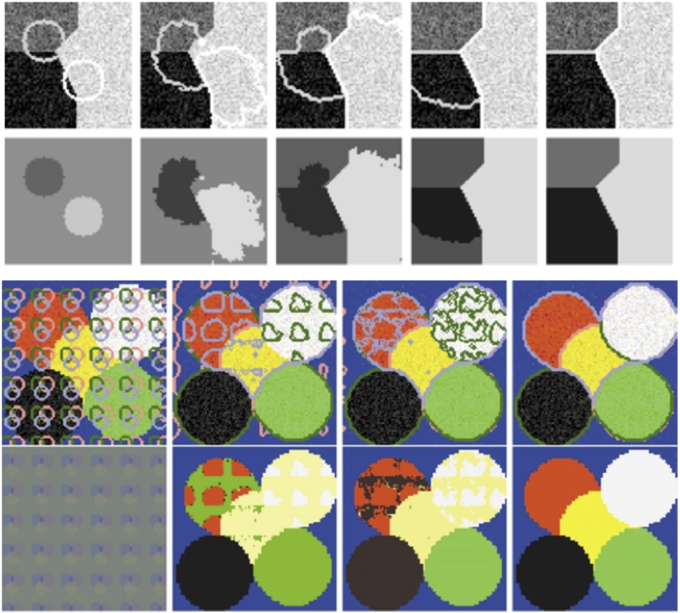
Level sets and multi-phase models (Chan, Vese, 2001, 2002)



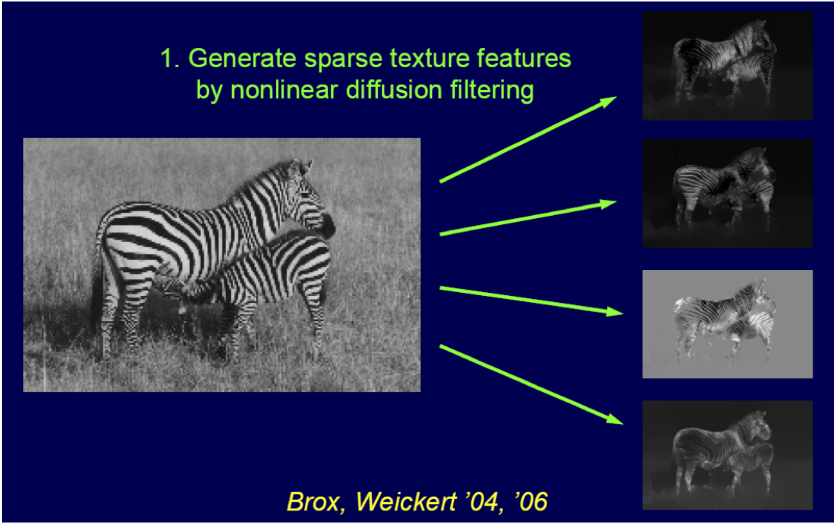
Segmentation of a MRI image with two level sets

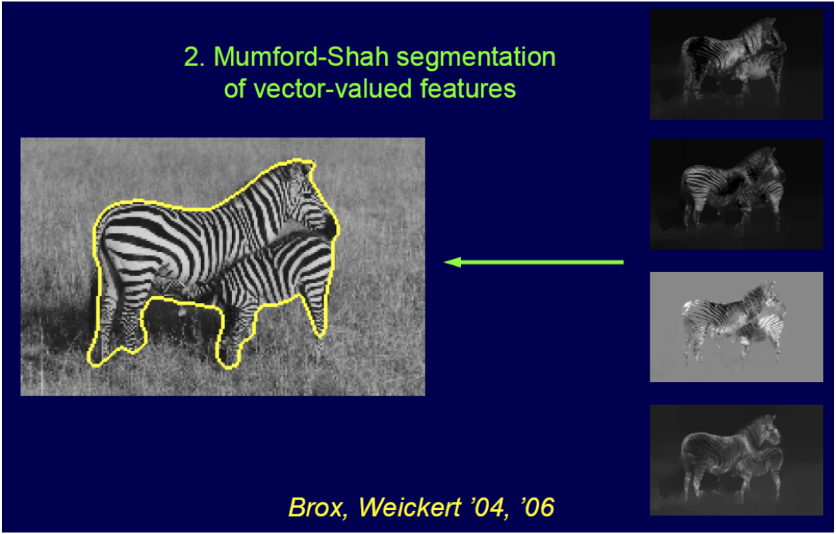


Example with junctions



Texture segmentation




2. Mumford-Shah segmentation
of vector-valued features
Brox, Weickert '04, '06

Region competition

Regions characterized by intensity distributions.
Probabilistic setting: partition $\mathcal{P}(\Omega) = \{\Omega_e, \Omega_i\}$ maximizing $p(I|\mathcal{P}(\Omega))p(\mathcal{P}(\Omega))$.

$$p(\mathcal{P}(\Omega)) \propto \nu \exp(-\nu|C|),\ \nu > 0$$

$$p(I|\mathcal{P}(\Omega)) = p(I|\Omega_e)p(I|\Omega_i) = \prod_{x\in\Omega_e} p_e(I(x),\theta_e) \prod_{x\in\Omega_i} p_i(I(x),\theta_i)$$

$$p(\mathcal{P}(\Omega)|I) = \nu \exp(-\nu|C|) \prod_{x\in\Omega_e} p_e(I(x),\theta_e) \prod_{x\in\Omega_i} p_i(I(x),\theta_i)$$

Formulation as an energy minimization problem:

$$E(\{C,\theta_e,\theta_i\}) = E_{reg}(C) + E_e(\{C,\theta_e\}) + E_i(\{C,\theta_i\})$$

$$\left\{ \begin{array}{l} E_{reg}(C) = -\log\nu + \nu|C|, \\ E_e(\{C,\theta_e\}) = -\int_{x\in\Omega_e} \log p_e(I(x),\theta_e)dx \\ E_i(\{C,\theta_i\}) = -\int_{x\in\Omega_i} \log p_i(I(x),\theta_i)dx \end{array} \right.$$

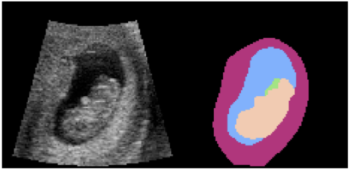
Implementation using level sets

$\phi : \Omega \to \mathbb{R}$, $\phi(x) > 0$ in Ω_e, $\phi(x) < 0$ in Ω_i and $\phi(x) = 0$ on C.

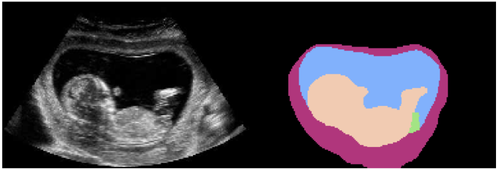
$$E(\phi, \theta_i, \theta_e) = E_{reg}(\phi) + E_e(\phi, \theta_e) + E_i(\phi, \theta_i)$$

$$\begin{cases} E_{reg}(\phi) = \nu \int_{x\in\Omega} \delta(\phi(x))|\nabla\phi(x)|dx, \\ E_e(\phi, \theta_e) = -\int_{x\in\Omega} H(\phi(x)) \log(p_e(I(x), \theta_e))dx \\ E_i(\phi, \theta_i) = -\int_{x\in\Omega} (1 - H(\phi(x))) \log(p_i(I(x), \theta_i))dx \end{cases}$$

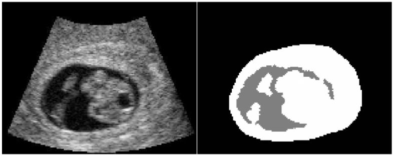
Example in ultrasound imaging (Jérémie Anquez)



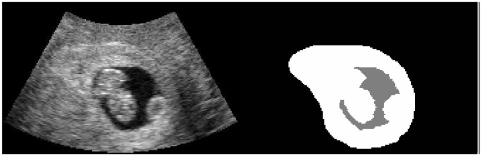
(a)



(b)

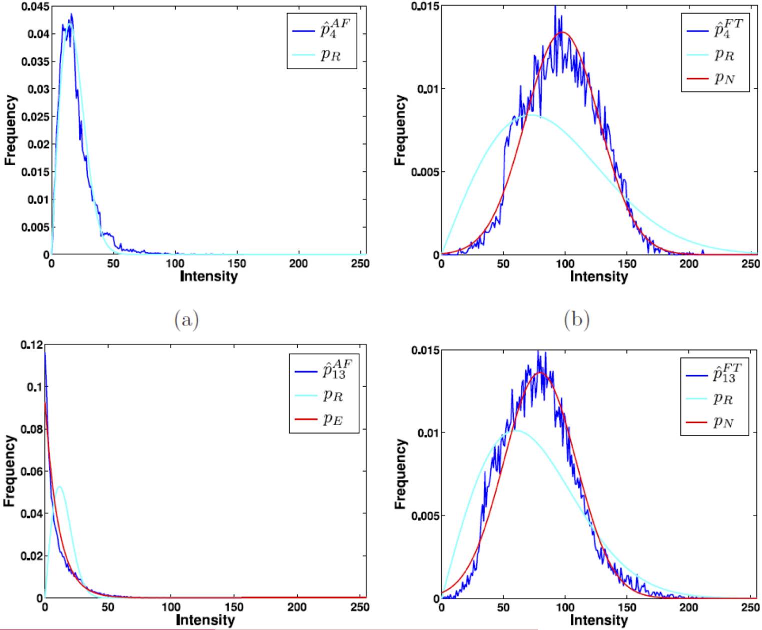


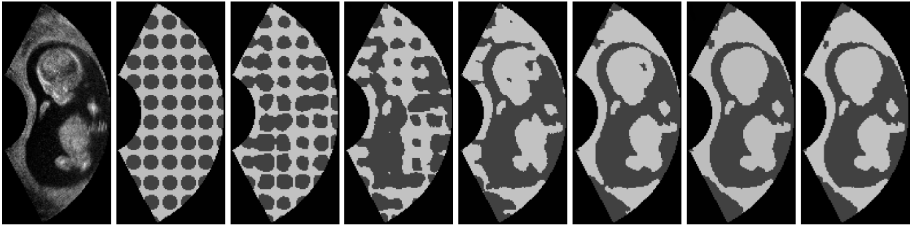
(c)



(d)

Learning distributions and their parameters:



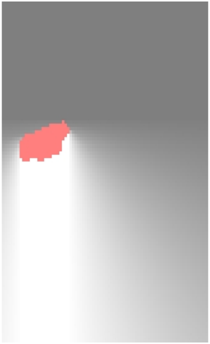


I_5 Init. 30 it. 60 it. 90 it. 120 it. 220 it. 390 it.

Constraining deformable models by spatial relations (Olivier Colliot et al.)

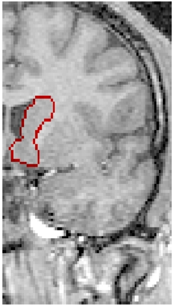
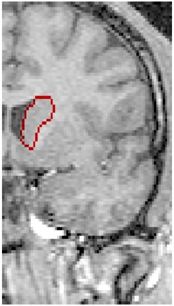
Examples

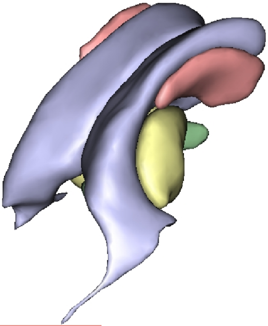
close to the lateral ventricle



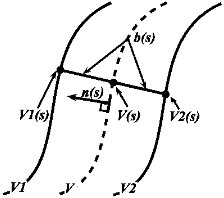
below

⇒ additional external force (avoids leaking in undesired regions)



Retina imaging (ISEP and XV-XX)



parallelism constraint

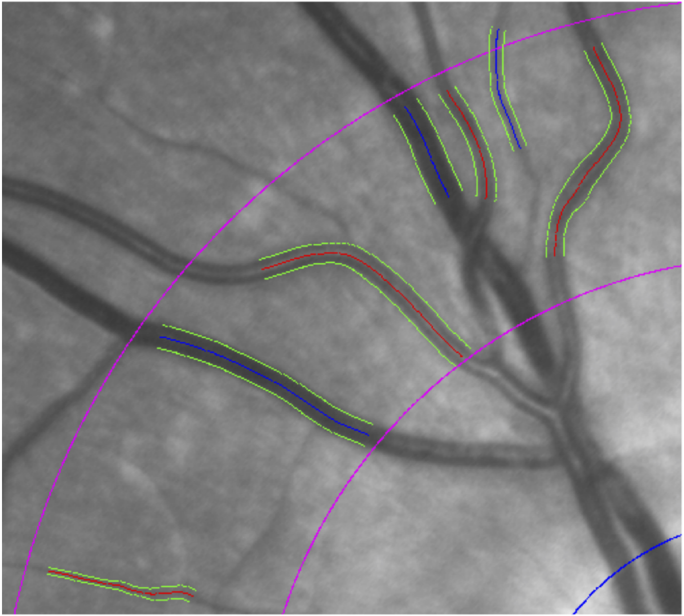
$$E(V, V_1, V_2, b) = E_{Image}(V_1) + E_{Image}(V_2) + E_{Int}(V) + R(V_1, V_2, b)$$

$$E_{Image}(V_i) = \int_0^1 P(V_i(s))ds = -\int_0^1 |\nabla I(V_i(s))|^2 \, ds$$

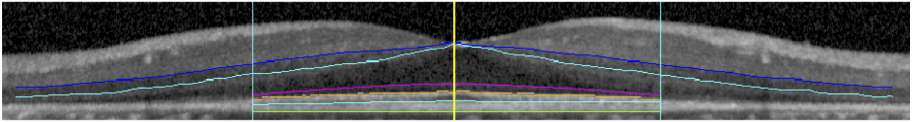
$$E_{Int}(V) = \frac{1}{2} \int \alpha(s) \left| \frac{\partial V(s,t)}{\partial s} \right|^2 + \beta(s) \left| \frac{\partial^2 V(s,t)}{\partial s^2} \right|^2 ds$$

$$R(V_1, V_2, b) = \int_0^1 \varphi(s)(b'(s))^2 ds$$

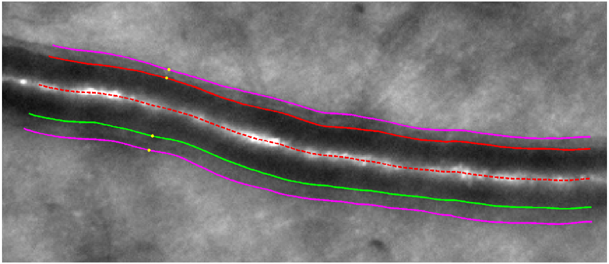
Eye fundus:



Optical coherence tomography (OCT):



Adaptive optics:



Practical work

Scikit-Image - Segmentation

- `skimage.segmentation.active_contour`: parametric representation, contour-based approach.
- `skimage.segmentation.chan_vese`: implicit representation (using level sets), region-based approach.

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