Exercises on Mathematical Morphology

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1 Properties of morphological operations

Are the following statements true or false?

		true	false
1	the erosion of a function is increasing with respect to the input function	◯	◯
2	the erosion of a function is increasing with respect to the structuring element	◯	◯
3	the erosion of a gray-level image fills "valleys"	◯	◯
4	an alternate sequential filter is increasing	◯	◯
5	an alternate sequential filter is idempotent	◯	◯
6	an alternate sequential filter is extensive	◯	◯
7	a convex set is invariant under closing by a compact convex structuring element	◯	◯
8	a convex set is invariant under opening by a compact convex structuring element	◯	◯
9	an erosion of size 3 followed by an erosion of size 2 is equivalent to an erosion of size 5	◯	◯
10	an opening of size 3 followed by an opening of size 2 is equivalent to an opening of size 5	◯	◯

2 Binary mathematical morphology

2.1 Morphological opening

Draw the opening of set X of Figure 1 with the structuring element B. Does the result depend on the origin? (indicated by a cross on the structuring element).

2.2 Selection of objects

A binary image contains disks of diameter 5, disks of diameter 10, and segments of length 5, 10 and 15, with various orientations. Which operations would allow:

- suppressing only disks of diameter 5?
- suppressing all disks (and only disks)?

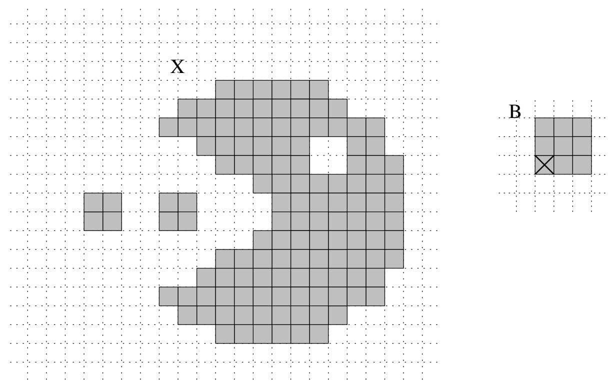


Figure 1: Operations on sets.

- selecting segments of length 15 whatever their orientation?

3 Mathematical morphology on functions (or gray level images)

Figure 2 illustrates a function f defined on a 1D space, and a structuring element B. Draw the result of the dilation of f by B, the erosion, the opening and the closing.

How many regional minima and regional maxima do f, $E(f, B)$ and f_B have?

How many peaks would be detected by a top-hat transform?

Draw the reconstruction of f from the marker g defined as $g(x) = \max(0, f(x) - 2)$. Interpret the result.

4 Relationship between binary and gray-level mathematical morphology

Let f be a function (representing a gray-level image) defined on $\mathbb{R}^n$. Its cuts (or level-sets) at height t are defined as:

$$f_t = \{x \in \mathbb{R}^n \mid f(x) \geq t\}.$$

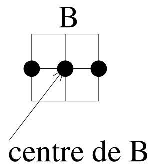
Establish the relationship between the dilated of the cut $D(f_t, B)$ and the cut of the dilation f, $D(f, B)_t$, where B is any binary structuring element.

5 Dilatation commutes with the supremum

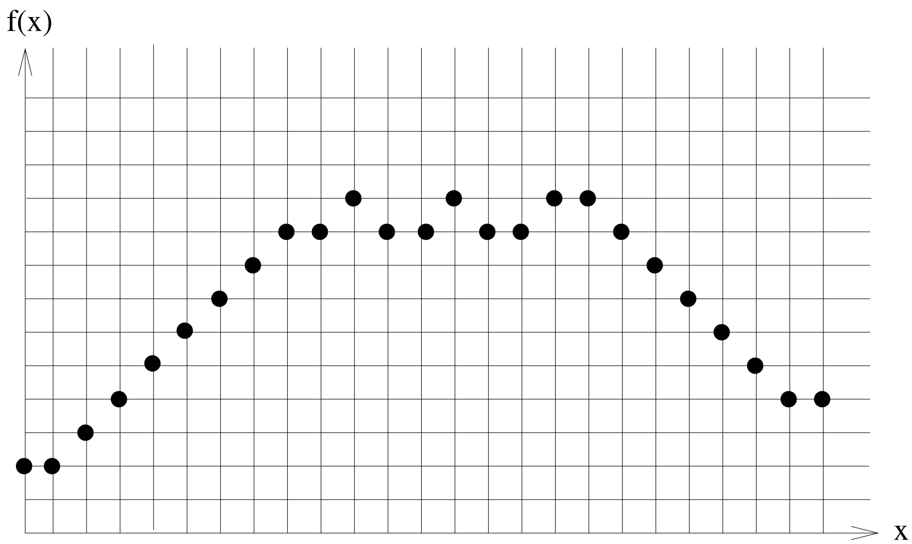
1. Prove that the dilation commutes with the supremum, i.e. for any functions f and g, and any set B, the following equation holds:

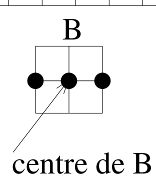
$$D(f \vee g, B) = D(f, B) \vee D(g, B)$$

where $f \vee g$ is defined by $\forall x, (f \vee g)(x) = \max[f(x), g(x)]$ (point-wise maximum).



élément structurant (segment horizontal centré)





élément structurant (segment horizontal centré)

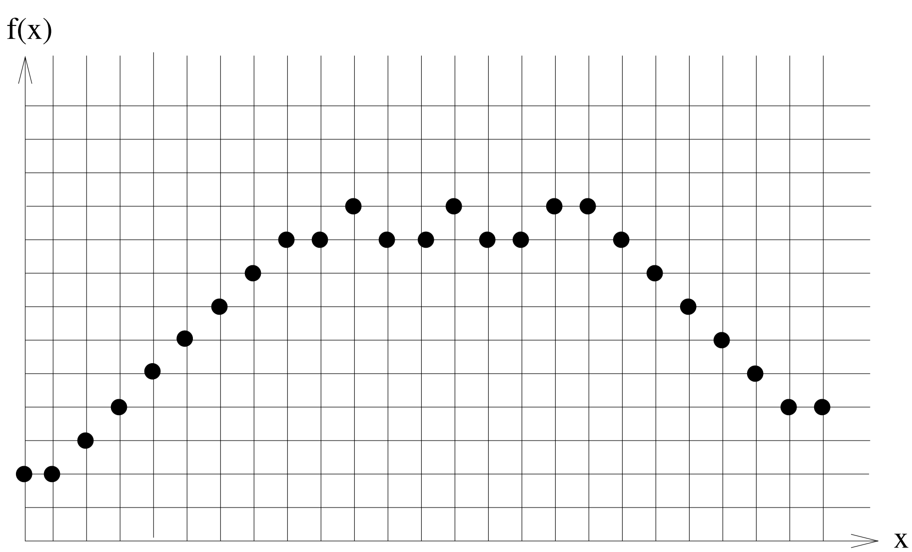


Figure 2: Function f and structuring element B.

2. Prove the following inequality for the infimum:

$$D(f \wedge g, B) \leq D(f, B) \wedge D(g, B).$$

Exhibit a simple counter-example where this inequality is strict.

6 Adjunctions and algebraic operators

Let $(\mathcal{T}, \leq)$ be a lattice ($\leq$ is a partial ordering such that $\forall (x, y) \in \mathcal{T}, \exists (x \vee y)$ and $\exists (x \wedge y)$ in $\mathcal{T}$). The lattice is complete if any family (finite or not) has a least upper bound and a greatest lower bound.

An algebraic dilation is an operator that commutes with the supremum of the lattice, and an algebraic erosion is an operator that commutes with the infimum of the lattice:

$$\forall (x_i) \in \mathcal{T},\ \delta(\vee_i x_i) = \vee_i \delta(x_i)$$

$$\forall (x_i) \in \mathcal{T},\ \varepsilon(\wedge_i x_i) = \wedge_i \varepsilon(x_i)$$

where i ranges over an arbitrary index set I.

A pair of operators (ε, δ) is an adjunction on $(\mathcal{T}, \leq)$ if:

$$\forall (x, y),\ \delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y)$$

1. Prove that if (ε, δ) is an adjunction, then for any x, x', y it holds that $\delta(x) \vee \delta(x') \leq y \Leftrightarrow \delta(x \vee x') \leq y$. Derive that δ is an algebraic dilation.
   It is possible to prove in a similar way that ε is an algebraic erosion.
2. Prove that δ is increasing.
3. Prove that
   - $\varepsilon\delta \geq Id$ and $\delta\varepsilon \leq Id$ (where Id is the identity function),
   - $\varepsilon\delta\varepsilon = \varepsilon$ and $\delta\varepsilon\delta = \delta$,
   - $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$ and $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$.

7 Algebraic opening

An algebraic opening on a lattice is an increasing, idempotent and anti-extensive operator. Let γ_1 and γ_2 be two algebraic openings. Prove the equivalence of the three following statements:

1. $\gamma_1 \leq \gamma_2$
2. $\gamma_1\gamma_2 = \gamma_2\gamma_1 = \gamma_1$
3. $Inv(\gamma_1) \subseteq Inv(\gamma_2)$ where Inv denotes the invariance domain.

Indication: prove that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. Recall that an opening γ can be expressed from its invariance domain as $\gamma(x) = \bigvee\{y \in Inv(\gamma),\ y \leq x\}$.

8 Morphological image analysis

Describe a potential segmentation method of the cells in Figure 3 using morphological operations. The result should be a binary image of the cells.

How to suppress cells that touch the border of the image?

How to separate connected cells?

How to select cells with a dark nucleus?

How to study the size distribution of the cells?

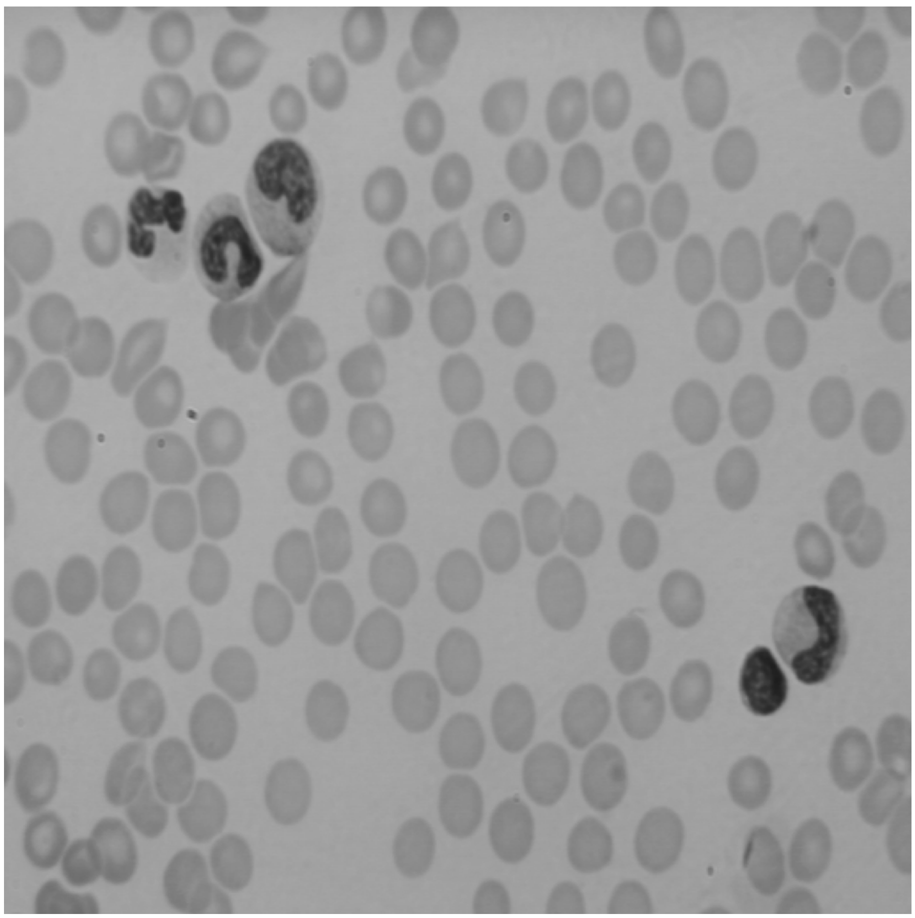


Figure 3: Biological cell image.