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Robust similarity between hypergraphs based on valuations and mathematical morphology operators

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ABSTRACT

This article aims at connecting concepts of similarity, hypergraph and mathematical morphology. We introduce new measures of similarity and study their relations with pseudo-metrics defined on lattices. More precisely, based on various lattices that can be defined on hypergraphs, we propose some similarity measures between hypergraphs based on valuations and mathematical morphology operators. We also detail new examples of these operators. The proposed similarity measures can be used in particular to introduce some robustness, up to some morphological operators. Some examples based on various dilations, erosions, openings and closings on hypergraphs illustrate the relevance of our approach. Potential applications to image comparison are suggested as well.

1. Introduction

In the field of automatic information processing, two aspects have gained importance in recent years. One consists in representing the structure of information, which is particularly crucial when dealing with large data, and the other is its increasing relation to algebraic processing methods relying on lattices.

The recent theory of hypergraphs takes its origin from the combinatorial set theory. First developed as a stand-alone mathematical theory, hypergraphs have become indispensable in many disciplines such as chemistry, physics, genetics, computer science, psychology… [30]. After having modeled data, an important task is the comparison of these data. Hence, the concept of similarity becomes very important. For instance, most of the disciplines cited above require the notions of comparison and similarity measures.

The notion of similarity plays a very important role in various fields of applied sciences. Classification is an example [8], and other examples such as indexing, retrieval or matching demonstrate the usefulness of the concept of similarity [9], with typical applications in image processing and image understanding. A recent trend in these domains is to rely on structural representations of the information (images for instance). Beyond the classical graph representations, and the associated notion of graph similarity, hypergraphs (in which edges can have any cardinality and are then called hyperedges), introduced in the 1960s [30], have recently proved useful. This concept has developed rapidly and has become both a powerful and well-structured mathematical theory for modeling complex situations. Let us consider the example of applications in image processing and understanding. Hypergraphs can be used to represent the structure of the image in different ways: vertices

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can be pixels and a hyperedge can be defined as a set of pixels sharing some properties, such as closeness in space or in a feature domain (color, texture…); it can also be defined at regional level, for representing spatial relations between regions or objects (which are then the vertices of the hypergraph) with arity greater than 2 (e.g. the "between" relation). In image applications, most similarity measures rely on features computed locally, or among the vertices of a hyperedge, and therefore do not completely exploit the structure of the hypergraph at this level [13,14,18,19].

Mathematical morphology offers interesting features to this purpose. Moreover, it has also strong algebraic bases, relying on the lattice theory [6,15,16,23–26].

In this paper we consider hypergraph representations of data, which can be endowed with a lattice structure, and mathematical morphology for manipulating these representations. Extending our preliminary work in [5], we propose new tools for defining similarity measures and metrics based on mathematical morphology. In order to deal with structured information, mathematical morphology has been developed on graphs [11,10,22,28,29], triangulated meshes [20], and more recently on simplicial complexes [12] and hypergraphs [3,4], where preliminary notions of dilation-based similarity were introduced. We propose to study similarities on lattices and more specifically on lattices of hypergraphs. We define some of them based on valuations on hypergraphs and mathematical morphology operators. They are illustrated on various types of lattices of hypergraphs. We also introduce new morphological operators, and show the interest of the proposed definitions for achieving robustness with respect to small variations of the compared hypergraphs. This robustness is intended in terms of indistinguishability, defined from morphological operators (i.e. two elements of a lattice are considered as indistinguishable if their images by a morphological operator are identical). Note that the paper is mainly theoretical, and although a few suggestions are provided to apply the proposed framework to images, such applications are out of the scope of the paper.

This paper is organized as follows. In Section 2 we recall some definitions on hypergraphs and lattices of hypergraphs on which morphological operators are defined. In Section 3, we show some general results on similarities, valuations and pseudo-metrics. Similarity and pseudo-metrics based on mathematical morphology are then defined in Section 4, with a number of illustrative examples.

2. Background and notations

Basic concepts on hypergraphs [1,7]: a *hypergraph* H denoted by $H = (V, E = (e_i)_{i\in I})$ is defined as a pair of a finite set V (vertices) and a family $(e_i)_{i\in I}$, (where I is a finite set of indices) of *hyperedges*. Hyperedges can be considered equivalently as subsets of vertices or as a relation between vertices of V. Let $(e_j)_{j\in\{1,2,\ldots,l\}}$ be a sub-family of hyperedges of E. The set of vertices belonging to these hyperedges is denoted by $v(\cup_{j\in\{1,2,\ldots,l\}} e_j)$, and $v(e)$ denotes the set of vertices forming the hyperedge e (note that $v(e) = e$ if hyperedges are considered as subsets of vertices). If $\cup_{i\in I} v(e_i) = V$, the hypergraph is without *isolated vertex* (a vertex x is isolated if $x \in V \setminus \cup_{i\in I} v(e_i)$). The set of isolated vertices is denoted by $V_{\setminus E}$. By definition the *empty hypergraph* is the hypergraph $H_\emptyset$ such that $V = \emptyset$ and $E = \emptyset$. We denote by $H(x)$ the star centered at x, for $x \in V$, i.e. the set of hyperedges containing x. A hypergraph is called *simple* if $\forall(i,j) \in I^2$, $v(e_i) \subseteq v(e_j) \Rightarrow i = j$. The *incidence graph* of a hypergraph $H = (V, E)$ is a bipartite graph $IG(H)$ with a vertex set $S = V \sqcup E$ (where $\sqcup$ stands for the disjoint union), and where $x \in V$ and $e \in E$ are adjacent if and only if $x \in v(e)$. Edges of a bipartite graph are considered as non directed. Conversely, to each bipartite graph $\Gamma = (V_1 \sqcup V_2, A)$, we can associate two hypergraphs: a hypergraph $H = (V, E)$, where $V = V_1$ and E contains the neighborhood of any element of V_2, where the neighborhood is considered in the bipartite graph Γ (i.e. $E = \{N_\Gamma(x) \mid x \in V_2\}$, where $N_\Gamma(x)$ is the neighborhood of a vertex in the graph Γ, derived from A), and its dual $H^* = (V^*, E^*)$, where $V^* = V_2$ and E^* is defined similarly.

Mathematical morphology on hypergraphs: in [4], we introduced mathematical morphology on hypergraphs. The first step was to define complete lattices on hypergraphs. Then the whole algebraic apparatus of mathematical morphology applies [6,15,16,24,26].

Let $(\mathcal{T}, \preceq)$ and $(\mathcal{T}', \preceq')$ be two complete lattices. All the following definitions and results are common to the general algebraic framework of mathematical morphology in complete lattices [6,15,16,24,26].

Definition 1. An operator $\delta : \mathcal{T} \to \mathcal{T}'$ is a dilation if: $\forall(x_i) \in \mathcal{T},\ \delta(\vee_i x_i) = \vee'_i \delta(x_i)$, where $\vee$ denotes the supremum associated with $\preceq$ and $\vee'$ the one associated with $\preceq'$.

An operator $\varepsilon : \mathcal{T}' \to \mathcal{T}$ is an erosion if: $\forall(x_i) \in \mathcal{T}',\ \varepsilon(\wedge'_i x_i) = \wedge_i \varepsilon(x_i)$, where $\wedge$ and $\wedge'$ denote the infimum associated with $\preceq$ and $\preceq'$, respectively.

An operator ψ on $\mathcal{T}$ is a morphological filter if it is increasing and idempotent. An anti-extensive filter is an algebraic opening and an extensive filter is an algebraic closing.

We denote the universe of hypergraphs by $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}$ the set of vertices (that we assume to be finite) and $\mathcal{E}$ the set of hyperedges. The powersets of $\mathcal{V}$ and $\mathcal{E}$ are denoted by $\mathcal{P}(\mathcal{V})$ and $\mathcal{P}(\mathcal{E})$, respectively. We denote a hypergraph by $H = (V, E)$ with $V \subseteq \mathcal{V}$ and $E \subseteq \mathcal{E}$. As developed in [4], several complete lattices can be built on $(\mathcal{V}, \mathcal{E})$. Let us denote by $(\mathcal{T}, \preceq)$ any of these lattices. We denote by $\wedge$ and $\vee$ the infimum and the supremum, respectively. The least element is denoted by $0_\mathcal{T}$ and the greatest element by $1_\mathcal{T}$. Here we will use three examples of complete lattices:

1. $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ (lattice over the power set of vertices);
2. $\mathcal{T}_2 = (\mathcal{P}(\mathcal{E}), \subseteq)$ (lattice over the power set of hyperedges);

3. $\mathcal{T}_3 = (\{H_j, j \in J\}, \preceq)$ where $\{H_j = (V_j, E_j), j \in J\}$ denotes a set of hypergraphs (for some index set J) defined on $(\mathcal{V}, \mathcal{E})$ such that $\forall j \in J, \forall e \in E_j, v(e) \subseteq V_j$, and the partial ordering is defined as [4]:
$$\forall (H_1 = (V_1, E_1), H_2 = (V_2, E_2)) \in \mathcal{T}_3^2, \quad H_1 \preceq H_2 \Leftrightarrow V_1 \subseteq V_2 \quad \text{and} \quad E_1 \subseteq E_2.$$
As shown in [4], we have $H_1 \wedge H_2 = (V_1 \cap V_2, E_1 \cap E_2)$ and $H_1 \vee H_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The smallest element is $0_{\mathcal{T}_3} = H_\emptyset = (\emptyset, \emptyset)$, and the greatest element is $1_{\mathcal{T}_3} = \mathcal{H} = (\mathcal{V}, \mathcal{E})$.

Examples of dilations on these lattices can be found in [4]. In Section 4, we provide further examples, along with the adjoint erosions, as well as examples of openings and closings.

As particular cases, note that if hyperedges contain at most two vertices, then the hypergraphs become classical graphs. Then the sublattice of $\mathcal{T}_3$ containing only graphs is the one considered in [10,11], and the mathematical morphology operators proposed in these references can then be used. Other particular cases are hypergraphs built from the hexagonal grid as in [21], or simplicial complexes as developed in [12]. Again, on the sublattice of $\mathcal{T}_3$ containing only such hypergraphs, the existing morphological operators can be used. Examples are also mentioned in [4]. While a simplicial complex is a particular hypergraph, it is also possible to associate a simplicial complex to each hypergraph $H = (V, E)$, from its hereditary hypergraph obtained by adding all possible hyperedges formed by non-empty subsets of vertices of $v(e)$, for all $e \in E$. Note that different hypergraphs can have the same hereditary hypergraph, and hence be associated with the same simplicial complex. However, if the hypergraph is simple, then there exists a unique simplicial complex associated with it via its hereditary hypergraph, and vice versa.

3. Similarity, valuation and pseudo-metric

3.1. Similarity and pseudo-metric

A similarity on a set $\mathcal{T}$ is defined as a function from $\mathcal{T} \times \mathcal{T}$ into $[0, 1]$ such that $\forall (x, y) \in \mathcal{T}^2, s(x, y) = s(y, x)$ and $s(x, x) = 1$. We will consider in particular the case where $\mathcal{T}$ is a lattice defined on hypergraphs. From a similarity s, a semi-pseudo-metric can be defined as $\forall (x, y) \in \mathcal{T}^2, d(x, y) = 1 - s(x, y)$. If moreover s satisfies $\forall (x, y, z) \in \mathcal{T}^3, s(x, z) + s(z, y) - 1 \leq s(x, y)$, then d is a pseudo-metric.[1] If the similarity s satisfies $\forall (x, y) \in \mathcal{T}^2, x \neq y \Rightarrow s(x, y) < 1$, then d is a semi-metric.

Proposition 1. *Let w be a monotonous (increasing) real positive valued function defined on a lattice $\mathcal{T}$ with $w(x) = 0 \Rightarrow x = 0_{\mathcal{T}}$. If $\forall (x, y, z) \in \mathcal{T}^3$ pairwise distinct and at least two of them being incomparable, $w(x \wedge z) + w(z \wedge y) \leq \min(w(x \vee z), w(z \vee y))$ then the function d defined as:*
$$d(x, y) = \begin{cases} 1 - \dfrac{w(x \wedge y)}{w(x \vee y)} & \text{if } w(x \vee y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$
is a pseudo-metric.

Note that the condition in this proposition is imposed only for pairwise distinct elements, and at least two of them being incomparable, since otherwise this would be satisfied only for trivial (constant) functions w.

Proof. The increasingness of w implies that d takes its values in $[0, 1]$. Moreover, the function w is symmetric and for all $x \in \mathcal{T}, d(x, x) = 0$.

Let us now prove the triangular inequality. For any $(x, y, z) \in \mathcal{T}^3$, we have the following cases:

- if $w(x \vee y) = 0$, then $d(x, y) = 0$ and $d(x, y) \leq d(x, z) + d(z, y)$;
- if $w(x \vee z) = 0$ or $w(z \vee y) = 0$, let us suppose that $w(z \vee y) = 0$, and then we have: $z = y = 0_{\mathcal{T}}$; hence $d(z, y) = 0$, $d(x, y) = d(x, z)$, and $d(x, y) \leq d(x, z) + d(z, y)$.
  Now let us assume that $w(x \vee y) \neq 0$, $w(x \vee z) \neq 0$ and $w(z \vee y) \neq 0$, for all following cases:
- let us consider the case where the three elements are comparable. Let us denote $h = d(x, z) + d(z, y) - d(x, y) = 1 - \frac{w(x \wedge z)}{w(x \vee z)} - \frac{w(z \wedge y)}{w(z \vee y)} + \frac{w(x \wedge y)}{w(x \vee y)}$ and let us show that $h \geq 0$. Since x and y play symmetrical roles, three cases need to be considered:
  - if $x \leq y \leq z$ $(\Rightarrow w(x) \leq w(y) \leq w(z)$ and $\frac{1}{w(x)} \geq \frac{1}{w(y)} \geq \frac{1}{w(z)})$, then $h = 1 - \frac{w(x)}{w(z)} - \frac{w(y)}{w(z)} + \frac{w(x)}{w(y)} = \frac{w(y)+w(x)}{w(y)} - \frac{w(x)+w(y)}{w(z)} = (w(x) + w(y))(\frac{1}{w(y)} - \frac{1}{w(z)}) \geq 0$;
  - if $x \leq z \leq y$, then $h = 1 - \frac{w(x)}{w(z)} - \frac{w(z)}{w(y)} + \frac{w(x)}{w(y)} = \frac{w(z)-w(x)}{w(z)} - \frac{w(z)-w(x)}{w(y)} = (w(z) - w(x))(\frac{1}{w(z)} - \frac{1}{w(y)}) \geq 0$;
  - if $z \leq x \leq y$, then $h = 1 - \frac{w(z)}{w(x)} - \frac{w(z)}{w(y)} + \frac{w(x)}{w(y)} = \frac{w(x)-w(z)}{w(x)} + \frac{w(x)-w(z)}{w(y)} = (w(x) - w(z))(\frac{1}{w(x)} + \frac{1}{w(y)}) \geq 0$;
- let us now consider the case where two elements are equal. Since $\forall x$, $d(x, x) = 0$, it is easy to show that in all cases $h \geq 0$;

[1] For a pseudo-metric, we have $d(x, x) = 0$ but we may have $d(x, y) = 0$ for $x \neq y$, and for a semi-metric the triangular inequality does not necessarily hold. So a semi-pseudo-metric satisfies $\forall (x, y) \in \mathcal{T}^2, d(x, y) = d(y, x), d(x, x) = 0$.

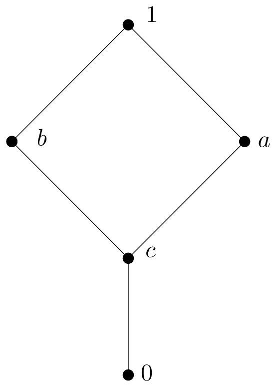

Fig. 1. An example of a lattice with a function w given by: $w(0) = 0$, $w(c) = 1$, $w(a) = w(b) = 2$, and $w(1) = 5$, which satisfies the hypothesis of Proposition 1: for all x, y, z in $\{0, 1, a, b, c\}$ pairwise distinct and with at least two elements incomparable, we have $w(x\wedge z)+w(z\wedge y) \leq \min(w(x\vee z), w(z\vee y))$.

- finally, let us consider the case where x, y, z are pairwise distinct and such that at least two of them are incomparable. Let $m = \min(w(x \vee z), w(z \vee y))$. We have:

$$\begin{aligned} h &= 1 - \frac{w(x \wedge z)}{w(x \vee z)} + 1 - \frac{w(z \wedge y)}{w(z \vee y)} - 1 + \frac{w(x \wedge y)}{w(x \vee y)} \\ &= 1 - \left(\frac{w(x \wedge z)}{w(x \vee z)} + \frac{w(z \wedge y)}{w(z \vee y)}\right) + \frac{w(x \wedge y)}{w(x \vee y)} \\ &\geq 1 - \left(\frac{w(x \wedge z) + w(z \wedge y)}{m}\right) + \frac{w(x \wedge y)}{w(x \vee y)}. \end{aligned}$$

By assumption $w(x \wedge z) + w(z \wedge y) \leq m$, hence $-\left(\frac{w(x\wedge z)+w(z\wedge y)}{m}\right) \in [-1, 0]$. We also have $\frac{w(x\wedge y)}{w(x\vee y)} \in [0, 1]$, and consequently

$$1 - \left(\frac{w(x \wedge z) + w(z \wedge y)}{m}\right) + \frac{w(x \wedge y)}{w(x \vee y)} \geq 0 \quad \text{and} \quad h \geq 0.$$

Therefore $d(x, y) \leq d(x, z) + d(z, y)$.

This proves that d is a pseudo-metric. ■

An example is illustrated in Fig. 1.

Corollary 1. *If the condition in Proposition 1 is restricted to either:*

- *if* $\forall(x, y, z) \in \mathcal{T}^3$ *pairwise distinct and at least two of them incomparable,* $w(x \wedge y) \geq w(x \wedge z) + w(z \wedge y)$ *and* $w(x \vee y) \leq \min(w(x \vee z), w(z \vee y))$,
- *or: if* $\forall(x, y, z) \in \mathcal{T}^3$ *pairwise distinct and at least two of them incomparable,* $w(x\wedge y) \geq w(x\wedge z)+w(z\wedge y)$ *and* $w(x\vee y) \leq w(x \vee z) + w(z \vee y)$,

then the function d from $\mathcal{T}^2$ *into* $[0, 1]$ *defined as:*

$$d(x, y) = \begin{cases} 1 - \dfrac{w(x \wedge y)}{w(x \vee y)} & \text{if } w(x \vee y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is a pseudo-metric.

Proof. The first case follows from the fact that the conditions imply the one in Proposition 1:

$$\begin{cases} w(x \wedge y) \geq w(x \wedge z) + w(z \wedge y) \\ w(x \vee y) \leq \min(w(x \vee z), w(z \vee y)) \end{cases}$$
$$\Rightarrow w(x \wedge z) + w(z \wedge y) \leq w(x \wedge y) \leq w(x \vee y) \leq \min(w(x \vee z), w(z \vee y)).$$

Hence d is a pseudo-metric. Note that for the first condition, the hypothesis that $w(x) = 0 \Rightarrow x = 0_{\mathcal{T}}$ is not necessary.

For the second case, let

$$h = d(x, z) + d(z, y) - d(x, y) = 1 - \frac{w(x \wedge z)}{w(x \vee z)} + 1 - \frac{w(y \wedge z)}{w(y \vee z)} - 1 + \frac{w(x \wedge y)}{w(x \vee y)}.$$

From the hypothesis, we have

$$h \geq 1 - \frac{w(x \wedge z)}{w(x \vee z)} - \frac{w(y \wedge z)}{w(y \vee z)} + \frac{w(x \wedge z) + w(z \wedge y)}{w(x \vee z) + w(z \vee y)}.$$

Since x and y play symmetrical roles, we can assume without loss of generality that $\frac{w(z \wedge y)}{w(z \vee y)} \geq \frac{w(x \wedge z)}{w(x \vee z)}$, and hence $\frac{w(x \wedge z)+w(z \wedge y)}{w(x \vee z)+w(z \vee y)} \geq \frac{w(x \wedge z)}{w(x \vee z)}$. So we have

$$h \geq 1 - \frac{w(x \wedge z)}{w(x \vee z)} - \frac{w(y \wedge z)}{w(y \vee z)} + \frac{w(x \wedge z)}{w(x \vee z)} = 1 - \frac{w(y \wedge z)}{w(y \vee z)}$$

and therefore $h \geq 0$. ■

Note that the conditions involved in this result are quite strong. In particular, they do not hold for simple valuations such as the cardinality on a graded lattice (see Section 3.3).

Proposition 2. *Under the conditions in Corollary 1 (still for pairwise distinct elements and at least two of them being incomparable), $d(x, y) = w(x \vee y) - w(x \wedge y)$ defines a pseudo-metric.*

Proof. We just have to verify the triangular inequality. We have for x, y, z pairwise distinct and with at least two of them incomparable:

$$w(x \vee y) \leq \min(w(x \vee z), w(z \vee y)) \Rightarrow w(x \vee y) \leq w(x \vee z) + w(z \vee y)$$
$$\Rightarrow w(x \vee y) - w(x \wedge y) \leq w(x \vee z) + w(z \vee y) - w(x \wedge y),$$

and from $w(x \wedge y) \geq w(x \wedge z) + w(z \wedge y)$:

$$w(x \vee y) - w(x \wedge y) \leq w(x \vee z) + w(z \vee y) - w(x \wedge z) - w(z \wedge y)$$

which shows that $d(x, y) \leq d(x, z) + d(z, y)$.

The case where two elements are equal is direct as in the proof of Proposition 1. The case where the elements are comparable is derived from $d(x, y) = \max(w(x), w(y)) - \min(w(x), w(y)) = |w(x) - w(y)|$ if x and y are comparable. ■

In the following, we will sometimes restrict the similarities to take values in $[0, 1] \cap \mathbb{Q}$ (i.e. rational values). All results of this section also apply under this restriction. This also applies to the next section.

3.2. Algebraic structure of similarities

In this section, we detail the structure of the collection of similarities. It is a linear space on the field with two elements. This allows us to consider a similarity as a vector. So we may decompose it as a linear combination. Since the collection of similarities forms a linear space, we can decompose any similarity over a base and thus express the similarity in a simpler way. In addition it also allows the use of linear algebra tools such as the notion of duality.

3.2.1. Lattice of similarities

Let us denote by I_n the $n \times n$ matrix where any coefficient is 1. Let s be a similarity measure on a set E. We may associate to it a matrix A called *similarity matrix* which is positive (i.e. all coefficients are positive), symmetric and has values 1 on its diagonal. We define the matrix $\widehat{A}$ associated with the similarity matrix A as:

$$\widehat{A} = A - I$$

where I is the identity matrix. If $A = \left(a_{i,j}\right)_{i,j \in 1,\dots,n}$ then

$$\widehat{A} = \begin{cases} \widehat{a}_{i,j} = a_{i,j} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Let A and B be two similarities matrices on the set E. The relation:

$$A \leq B \Leftrightarrow 0 \leq \widehat{B} - \widehat{A} \Leftrightarrow 0 \leq \widehat{b}_{i,j} - \widehat{a}_{i,j}, \quad \text{for all } i, j \tag{1}$$

is a partial order. In the same way we define a strict partial order as:

$$\begin{cases} A < B \Leftrightarrow 0 < \widehat{B} - \widehat{A} \Leftrightarrow & 0 \leq \widehat{b}_{i,j} - \widehat{a}_{i,j}, \text{ for all } i, j, \\ & \text{and } \exists (k, l) \mid 0 < \widehat{b}_{k,l} - \widehat{a}_{k,l}. \end{cases} \tag{2}$$

Define now

$$A \vee B = I + \widehat{A} \vee \widehat{B} = I + \left(\max(\widehat{a}_{i,j}, \widehat{b}_{i,j})\right)_{i,j \in 1,\dots,n}, \tag{3}$$

and

$$A \wedge B = I + \widehat{A} \wedge \widehat{B} = I + \left(\min(\widehat{a}_{i,j}, \widehat{b}_{i,j})\right)_{i,j \in 1,\ldots,n}. \tag{4}$$

Note that $A \wedge B$ and $A \vee B$ are similarity matrices. We have:

$$\widehat{A} \vee \widehat{B} = \widehat{A \vee B}, \quad \text{and} \quad \widehat{A} \wedge \widehat{B} = \widehat{A \wedge B}.$$

Proposition 3. *Let $\mathcal{S}$ be the collection of similarities on a set E. This set, equipped with the relation $\leq$, is a complete lattice $(\mathcal{S}, \leq)$. The supremum and infimum are $\vee$ (Eq. (3)) and $\wedge$ (Eq. (4)), respectively. The smallest element is I and greatest element is I_n.*

Proof. The matrices $C = A \vee B$ and $D = A \wedge B$ are similarity matrices. Indeed, on the diagonal of C and D all values are equal to 1; these two matrices are symmetric and all coefficients $c_{i,j}$ and $d_{i,j}$ belong to $[0, 1]$.

The other results can be derived directly from the fact that the defined order is the marginal (Pareto) order on the non-diagonal elements of the matrices. Hence the minimum and maximum are simply obtained component wise, and the smallest (respectively greatest) element has 0 (respectively 1) everywhere, except on the diagonal where the components are always equal to 1. ∎

3.2.2. Group of similarities

For all $(A, B, C) \in \mathcal{S}^3$, we define the following operation:

$$A \oplus B = C \Leftrightarrow \hat{A} + \hat{B} \equiv \hat{C} (\text{mod } 1)$$

where $\hat{A} + \hat{B} \equiv \hat{C}$ (mod 1) means that $\hat{a}_{i,j} + \hat{b}_{i,j} \equiv \hat{c}_{i,j}$ (mod 1) for all $i, j \in \{1, \ldots, n\}$.

Proposition 4. *Let $\mathcal{S}$ be the collection of similarities on a set E (with $|E| = n$). The set $\mathcal{S}$ equipped with the operation $\oplus$ is a commutative group $(\mathcal{S}; \oplus)$.*

Proof. Since $\hat{A} + \hat{B} \equiv \hat{C}$ (mod 1), for all $i, j \in \{1, \ldots, n\}$, $\hat{c}_{i,j} \in [0, 1]$, for all $i, j \in \{1, \ldots, n\}$; hence, the matrix C is completely determined and $C \in \mathcal{S}$. So the operation $\oplus$ is internal.

It is easy to verify that the operation $\oplus$ is associative and commutative.

Now:

$$\hat{A} + \hat{I} = \hat{I} + \hat{A} = \hat{A} + 0 = 0 + \hat{A} = \hat{A} \Leftrightarrow A \oplus I = I \oplus A = A.$$

Moreover:

$$\hat{A} + (-\hat{A}) = (-\hat{A}) + \hat{A} = 0 = \hat{I} \Leftrightarrow A \oplus (-A) = (-A) \oplus A = I. \quad ∎$$

Consequently $\mathcal{S}$ is a linear space on the field $\mathbb{F}_2$.

3.3. Valuation on a lattice $(\mathcal{T}, \preceq)$ and pseudo-metric

In this section, we build pseudo-metrics in a similar way as in Section 3.1, but now from specific functions w, namely valuations on the considered lattices, which are a common and useful notion in the lattice framework. This will constitute the basis of our proposals in Section 4. This choice is motivated by the fact that the condition for w being a valuation is weaker than the conditions on w in Section 3.1, and concrete useful examples can be more easily exhibited.

Definition 2 *([2])*. A valuation w on a lattice $(\mathcal{T}, \preceq)$ is defined as a real-valued function such that: $\forall (x, y) \in \mathcal{T}^2$, $w(x) + w(y) = w(x \wedge y) + w(x \vee y)$. A valuation is increasing if $\forall (x, y) \in \mathcal{T}^2$, $x \preceq y \Rightarrow w(x) \leq w(y)$.

In the following we consider only increasing valuations. We have then $\forall x \in \mathcal{T}$, $w(0_{\mathcal{T}}) \leq w(x) \leq w(1_{\mathcal{T}})$, and $\forall (x, y) \in \mathcal{T}^2$, $w(x \wedge y) \leq w(x \vee y)$. A pseudo-metric can be derived as follows [2].

Theorem 1 *([2] (Theorem 1 in Chapter X))*. *Let w be an increasing valuation on $(\mathcal{T}, \preceq)$. Then d, defined by $\forall (x, y) \in \mathcal{T}^2$, $d(x, y) = w(x \vee y) - w(x \wedge y)$ is a pseudo-metric. The following inequality also holds: $\forall (a, x, y) \in \mathcal{T}^3$, $d(a \vee x, a \vee y) + d(a \wedge x, a \wedge y) \leq d(x, y)$.*

Note that this result requires weaker assumptions than the conditions in Proposition 2.

As an example, let us consider the lattice $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ [4]. The cardinality defines an increasing valuation: $\forall V \subseteq \mathcal{V}$, $w(V) = |V|$. We have $w(V) = 0 \Leftrightarrow V = \emptyset = 0_{\mathcal{T}}$ and $w(1_{\mathcal{T}}) = |\mathcal{V}|$. In this case, d is a metric (in particular we have $d(V, V') = 0 \Leftrightarrow V = V'$), and can be expressed as:

$$\forall (V, V') \in \mathcal{P}(\mathcal{V})^2, \quad d(V, V') = |V \cup V'| - |V \cap V'| = |V| + |V'| - 2|V \cap V'| = |V \Delta V'|.$$

Proposition 5. *The lattice $\mathcal{T}_3 = (\{H_j, j \in J\}, \preceq)$ is distributive (and hence modular).*

Proof. Let $H_1 = (V_1, E_1), H_2 = (V_2, E_2), H = (V, E)$ be any three hypergraphs in $\mathcal{T}_3$. We have

$$\begin{aligned} H_1 \wedge (H_2 \vee H_3) &= H_1 \wedge (V_2 \cup V_3, E_2 \cup E_3) = (V_1 \cap (V_2 \cup V_3), E_1 \cap (E_2 \cup E_3)) \\ &= ((V_1 \cap V_2) \cup (V_1 \cap V_3), (E_1 \cap E_2) \cup (E_2 \cup E_3)) \\ &= (H_1 \wedge H_2) \vee (H_1 \wedge H_3). \end{aligned}$$

Similarly we have $H_1 \vee (H_2 \wedge H_3) = (H_1 \vee H_2) \wedge (H_1 \vee H_3)$. ■

Note that if the hypergraphs are supposed to be without isolated vertices, the double partial ordering reduces to inclusion between hyperedge sets and $\mathcal{T}_3$ is isomorphic to $\mathcal{T}_2 = (\mathcal{P}(\mathcal{E}), \subseteq)$ and hence distributive (and a fortiori modular). Here, we consider the more general case where V can contain isolated vertices.

Based on results on the height function in [2], an interesting property is that the height function on $\mathcal{T}_3$ defines a strictly monotonous valuation w, and, $\forall (x, y) \in \mathcal{T}_3^2$, if y covers x (i.e. $x \prec y$ and $\nexists z \in \mathcal{T}_3, x \prec z \prec y$) then $w(y) = w(x) + 1$. It is easy to show that H' covers H iff H' differs from H only by an ''elementary'' transformation, which can be of two types:

- either $V' = V \cup \{v\}$ for some $v \in \mathcal{V}$, and $E' = E$;
- or $V' = V$ and $E' = E \cup \{e\}$ for some e such that $v(e) \subseteq V$.

Proposition 6. *On $\mathcal{T}_3 = (\{H_j, j \in J\}, \preceq)$, the valuation defined by the height function is equal to: $\forall H = (V, E) \in \mathcal{T}_3$, $w(H) = |V| + |E|$.*

Proof. From the characterization of the covering relation, the height of (V, E) is simply the number of elementary transformations required to transform $0_{\mathcal{T}_3} = (\emptyset, \emptyset)$ into (V, E), i.e. $|V| + |E|$. ■

A simple example is illustrated in Fig. 2, for $\mathcal{V} = \{v, v'\}$ and $\mathcal{E} = \mathcal{P}(\mathcal{V})$. The hypergraph $(\{v\}, \emptyset)$ is deduced from $0_{\mathcal{T}_3} = (\emptyset, \emptyset)$ by an elementary transformation of type (i), while the hypergraph $(\{v\}, e = \{v\})$ is deduced from $(\{v\}, \emptyset)$ by an elementary transformation of type (ii). We have for instance $w((\{v\}, e = \{v\})) = 2$ and $w((\{v, v'\}, e = \{v\})) = 3$.

4. Mathematical morphology and similarity between hypergraphs

4.1. Similarity and dilation

The following result establishes a first link between similarity and dilation, exhibiting a dilation on hypergraphs for some particular similarity functions.

Theorem 2. *Let S be a set of cardinality m and s a similarity on S such that $s(x) \in [0, 1] \cap \mathbb{Q}$, for all $x \in S$. Let us assume that s can be written as:*

$$\forall (u_i, u_j) \in S^2,\ i, j \in \{1, \dots, m\}, \quad s(u_i, u_j) = \left(\frac{x_{i,j}}{x_i + x_j - x_{i,j}} \right)$$

with $\forall i, j, x_i \in \mathbb{N}, x_{i,j} \in \mathbb{N}, x_{i,j} = x_{j,i}, x_{i,i} = x_i$ and $x_i \geq \sum_{j=1, j\neq i}^{m} x_{i,j}$. Then there is a hypergraph $H = (V, E)$ with $|E| = m$ and a dilation from $(\mathcal{P}(E), \subseteq)$ into $(\mathcal{P}(V), \subseteq)$ such that:

$$s(u_i, u_j) = \frac{|\delta(e_i) \cap \delta(e_j)|}{|\delta(e_i) \cup \delta(e_j)|}, \quad \textit{for all } i, j \in \{1, \dots, m\}. \tag{5}$$

Proof. We construct a bipartite graph $\Gamma = (V_1 \sqcup V_2, A)$ (where $\sqcup$ denotes the disjoint union) by induction on m in the following fashion:

- if $m = 1$: then we have only one similarity value which is equal to 1, and we can define a hypergraph with only one non-empty hyperedge e and $\delta(e) = v(e)$ (whatever the vertices of e are), leading to $\frac{|\delta(e)\cap\delta(e)|}{|\delta(e)\cup\delta(e)|} = 1$;
- if $m = 2$.
  1. we define $V_1 = \{e_1, e_2\}$;
  2. we define V_2 in the following way: we first put $x_{1,2}$ elements in $V_2 : \{v_{1,2}^1, v_{1,2}^2, \dots, v_{1,2}^{x_{1,2}}\} \subseteq V_2$. Then:
     (a) if $x_{1,2} = x_1$ and $x_{1,2} = x_2$, then $V_2 = \{v_{1,2}^1, v_{1,2}^2, \dots, v_{1,2}^{x_{1,2}}\}$;
     (b) if $x_{1,2} < x_1$ and $x_{1,2} < x_2$, then we add to V_2 some more elements: $Z_1 = \{z_1^1, z_1^2, \dots, z_1^{x_1 - x_{1,2}}\}$ and $Z_2 = \{z_2^1, z_2^2, \dots, z_1^{x_2 - x_{1,2}}\}$;
     (c) if $x_{1,2} < x_1$ and $x_{1,2} = x_2$ (respectively $x_{1,2} = x_1$ and $x_{1,2} < x_2$), then we add only Z_1 (respectively Z_2).

  Now we construct A:
  1. we put an edge from e_1 to each $v_{1,2}^k,\ k \in \{1, \dots, x_{1,2}\}$, and similarly an edge from e_2 to each $v_{1,2}^k$;
  2. we put an edge from e_1 to z_1^l, for all $l \in \{1, \dots, x_1 - x_{1,2}\}$ if $Z_1 \neq \emptyset$;
  3. we put an edge from e_2 to z_2^h, for all $h \in \{1, 2, \dots, x_2 - x_{1,2}\}$ if $Z_2 \neq \emptyset$.

  Hence, we obtain a bipartite graph with $|V_1| = 2, |V_2| = x_{1,2} + x_1 - x_{1,2} + x_2 - x_{1,2} = x_1 + x_2 - x_{1,2}$. This graph is the incidence graph of a hypergraph (V, E) with $V = V_2, E = V_1, |e_1| = x_1, |e_2| = x_2$ and $|e_1 \cap e_2| = x_{1,2}$.

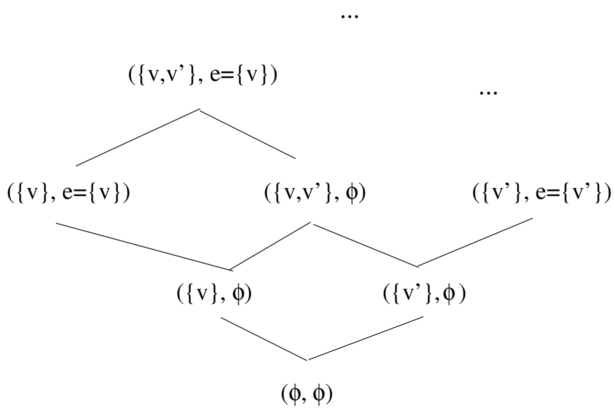

Fig. 2. An excerpt of the lattice $\mathcal{T}_3$ of hypergraphs for $\mathcal{V} = \{v, v'\}$ and $\mathcal{E} = \mathcal{P}(\mathcal{V})$.

Next, we define a dilation on singletons as $\delta(\{e_1\}) = v(e_1)$ and $\delta(\{e_2\}) = v(e_2)$, and on any subset using the sup-generating property and the commutativity of any dilation with the supremum. We have $\forall(i,j) \in \{1,2\} \times \{1,2\}$, $|\delta(e_i) \cap \delta(e_j)| = x_{i,j}$ and $|\delta(e_i) \cup \delta(e_j)| = x_i + x_j - x_{i,j}$.

- Assume than the assertion is true for $|S| = m - 1$.
  Let s be a similarity on S with $|S| = m$ and verifying the above properties. By removing from the matrix $\big(s(u_i, u_j)\big)_{i,j}$ the last column and the last row, we obtain a similarity s' on a set with $m - 1$ elements and satisfying the properties of our assertion. By the induction hypothesis, there is therefore a hypergraph $H' = (V', E')$ and a dilation δ' from $(\mathcal{P}(E'), \subseteq)$ into $(\mathcal{P}(V'), \subseteq)$ such that:

$$s'(u_i, u_j) = \frac{|\delta'(e_i) \cap \delta'(e_j)|}{|\delta'(e_i) \cup \delta'(e_j)|}, \quad \text{for all } i, j \in \{1, \dots, m-1\}.$$

  Let $\Gamma' = (V'_1 \sqcup V'_2, A')$ be the bipartite graph associated with the hypergraph $H' = (V', E')$.
  1. We define $V_1 = V'_1 \sqcup \{e_m\}$.
  2. For all $e_i \in V'_1$ with $v(e_i) = \{y_1, y_2, \dots, y_{|e_i|=x_i}\}$, we add edges from y_j, $j \in \{1, \dots, x_{j,m}\} \subseteq \{1, \dots, |e_i| = x_i\}$. This is always possible since $x_i \geq x_{i,m}$ and $x_m \geq \sum_{j=1}^{m-1} x_{m,j}$. If $x_m > \sum_{j=1}^{m-1} x_{m,j}$, then we add $x_m - \sum_{j=1}^{m-1} x_{m,j}$ new vertices to V'_2 to build V_2 (otherwise $V_2 = V'_2$) and we add edges from all vertices of V_2 to e_m. Hence we have x_m edges from e_m.
  3. We obtain a new bipartite graph $\Gamma = (V_1 \sqcup V_2, A)$ which is the incidence graph of a hypergraph $H = (V, E)$.

  Let $\delta(\{e_m\}) = v(e_m)$, and $\delta(\{e_i\}) = \delta'(\{e_i\})$ for $i < m$. This defines a dilation on singletons, from which a dilation from $(\mathcal{P}(E), \subseteq)$ into $(\mathcal{P}(V), \subseteq)$ is derived using the sup-generating property and the commutativity with the supremum. By construction, we have $\forall(i,j) \in \{1, \dots, m\} \times \{1, \dots, m\}$, $|\delta(e_i) \cap \delta(e_j)| = x_{i,j}$ and $|\delta(e_i) \cup \delta(e_j)| = x_i + x_j - x_{i,j}$. ■

This result establishes a first link between similarities and dilations. The importance of similarity has largely been highlighted on graphs, which are a special case of hypergraphs, and it can then be extended to hypergraphs. All similarities that can be expressed as in Theorem 2, such as the Jaccard index (ratio of cardinalities of intersection and union) for instance [17], can be linked to a dilation and a hypergraph. In the next sections, we will go one step further and define similarities from various morphological operators.

Let us detail an example, illustrating the link between hypergraphs, graphs and simplicial complexes under particular conditions, and using the construction proposed in the proof of Theorem 2. We start with the following lemma.

Lemma 1. *Let $H = (V, E)$ be a hypergraph. Its dual $H^* = (E, (H(x))_{x \in V})$ is a graph if and only if no vertex belongs to more than two hyperedges.*

Proof. Assume that the dual of H is a graph. The hyperedges of H^* being the stars of H, they have a cardinality equal at most to 2. Consequently any vertex of H has a degree equal to either 1 or 2. Conversely, assume that no vertex lies in more than two hyperedges. For all $x \in V$, $|H(x)| \leq 2$. Hence H^* is a graph. ■

Let $H = (V, E)$ be a hypergraph. A *star center* is a subset C of V such that for all $x, y \in C$, $H(x) = H(y)$. The relation R defined as $xRy \Leftrightarrow H(x) = H(y)$ is an equivalence relation which defines a partition of V (whose elements are star centers). Moreover for all $e \in E$, this relation provides a partition of $v(e)$.

Theorem 3. *Let $\Gamma = (V, E)$ be a graph, then its dual $H^* = (V^*, E^*)$ verifies:*

$$\forall e_i^* \in E^*, \quad |e_i^*| \geq \sum_{j=1, j \neq i}^{m} |e_i^* \cap e_j^*|, \tag{6}$$

where m denotes the number of hyperedges of H^*. Conversely let $H = (V, E)$ be a hypergraph verifying Eq. (6) and constructed as in the proof of Theorem 2, then its dual is a graph $\Gamma^* = (V^*, E^*)$.

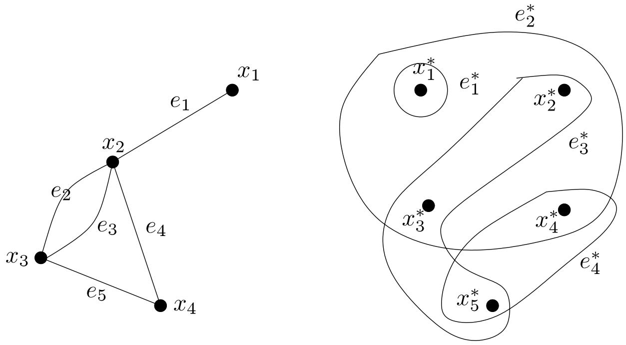

Fig. 3. A graph H (left) and its dual H^* (right). It is easy to see that H^* verifies Eq. (6).

Proof. Let $\Gamma = (V, E)$ be a graph and $H^* = (V^*, E^*)$ its dual. Let $e_i^* \in E^*$. From the equivalence relation R, $v(e_i^*) = \sqcup_{x \in v(e_i^*)} C_x$, where C_x is the class containing x. From Lemma 1, for all $x \in v(e_i^*)$, $|H(x)| \leq 2$. Hence,

$$|e_i^*| = \sum_{x \in v(e_i^*)} |C_x| = \sum_{\substack{x \in v(e_i^*) \\ x \in v(e_j^*), j \neq i}} |C_x| + |\mathcal{A}| \geq \sum_{j=1, j \neq i}^{m} |e_i^* \cap e_j^*|$$

where

$$\mathcal{A} = \{x \in v(e_i^*) \mid x \notin v(e_j^*), j \neq i\}.$$

Note that since Γ is a graph, for a fixed i, all intersections $e_i^* \cap e_j^*$, for $j = 1 \dots m, j \neq i$, are distinct subsets of e_i^*, hence the result.

Conversely, let $H = (V, E)$ be a hypergraph verifying Eq. (6) and constructed as in the proof of Theorem 2. It is easy to verify that no vertex of H lies in more than two hyperedges, so for all $x \in V$, $|H(x)| \leq 2$ and the dual of H is a graph. ∎

An example is illustrated in Fig. 3.

Corollary 2. *Let $\Gamma = (V, E)$ be a graph. An operator δ is a dilation on Γ if and only if the dual operator δ^* is a dilation on Γ^**

This result follows directly from the results in Section 5 of [4].

Let us now consider graphs defined over $\mathcal{V}$ and the lattice of the powerset of the edge sets. The duals are hypergraphs as above. Let us consider a positive real valued function m on the vertex set. Let us define a function w on hyperedge sets as:

$$w(E) = \sum_{e \in E} \sum_{v_i \in v(e)} m(v_i).$$

This function w is increasing, and we have:

$$\forall E_1, E_2, E_3, \quad w(E_1 \cap E_2) + w(E_2 \cap E_3) \leq w(E_2) \leq \min(w(E_1 \cup E_2), w(E_2 \cup E_3)).$$

Therefore w satisfies the condition of Proposition 1. Note that an alternative definition for w could be $w(E) = \sum_{v_i \in \mathcal{V} \mid \exists e \in E, v_i \in v(e)} m(v_i)$, which would lead to similar results.

A similar construction applies for simplicial complexes. Starting from a hypergraph satisfying Eq. (6), a simplicial complex can then be derived from its hereditary hypergraph, as mentioned in Section 2. The lattice structure is then the same as in [12], and w can be defined from a positive real valued function m as above. This function w satisfies the condition of Proposition 1.

4.2. Similarity from a valuation and a morphological operator

Let us consider any lattice of hypergraphs $(\mathcal{T}, \preceq)$, an increasing valuation w and a morphological operator ψ defined on this lattice. In this section, we generalize ideas suggested in [4] in the particular case where the lattice was the power set of vertices, w was the cardinality and ψ was a dilation.

Definition 3. Let $(\mathcal{T}, \preceq)$ be a lattice, w an increasing valuation on this lattice such that $w(x) = 0$ iff $x = 0_{\mathcal{T}}$, and ψ an increasing operator from $(\mathcal{T}, \preceq)$ into $(\mathcal{T}, \preceq)$ such that $\psi(x) = 0_{\mathcal{T}} \Rightarrow x = 0_{\mathcal{T}}$. We define a real-valued function s as: $\forall (x, y) \in \mathcal{T}^2 \setminus (0_{\mathcal{T}}, 0_{\mathcal{T}})$, $s(x, y) = \frac{w(\psi(x) \wedge \psi(y))}{w(\psi(x) \vee \psi(y))}$ and $s(0_{\mathcal{T}}, 0_{\mathcal{T}}) = 1$.

In this work, we deal with the particular case of lattices of hypergraphs, and of morphological increasing operators, such as dilations, erosions, openings, closings, other filters.

Proposition 7. *The function s introduced in Definition 3 is a similarity.*

Proof. s is well defined, thanks to the conditions imposed on w and ψ. Since w is increasing, we have $\forall(x, y) \in \mathcal{T}^2$, $s(x, y) \in [0, 1]$. Moreover s is symmetrical (by construction) and $\forall x \in \mathcal{T}$, $s(x, x) = \frac{w(\psi(x)\wedge\psi(x))}{w(\psi(x)\vee\psi(x))} = 1$. ■

Note that this result extends to morphological operators from a lattice $(\mathcal{T}, \preceq)$ into a different one $(\mathcal{T}', \preceq)$.

In the particular case where ψ is a dilation and w is the cardinality, then the similarity has the same expression as in Eq. (5) and therefore this result provides a kind of reversed construction with respect to the one in Theorem 2. In this theorem, from a similarity having a specific form we derived a hypergraph and a dilation. Here we start from a dilation and derive a similarity.

In a similar way as in Theorem 1, we introduce a pseudo-metric defined from w and ψ.

Proposition 8. *Let w and ψ defined on $(\mathcal{T}, \preceq)$ as in Definition 3. The real-valued function d_ψ defined as: $\forall(x, y) \in \mathcal{T}^2$, $d_\psi(x, y) = w(\psi(x) \vee \psi(y)) - w(\psi(x) \wedge \psi(y))$ is a pseudo-metric.*

Proof. Since $d_\psi(x, y) = d(\psi(x), \psi(y))$ for d defined from w as in Theorem 1, the properties of d_ψ are straightforwardly derived from the ones of d, and d_ψ is a pseudo-metric. ■

Note that again this result requires weaker assumptions than the conditions in Proposition 2.

In the particular case where $\mathcal{T}$ is the power set of the set of vertices or of hyperedges (with $\preceq$ equal to $\subseteq$), and the valuation is the cardinality, then d_ψ is a metric if ψ is injective (indeed $d_\psi(x, y) = 0 \Rightarrow |\psi(x) \cup \psi(y)| - |\psi(x) \cap \psi(y)| = 0 \Rightarrow \psi(x) = \psi(y) \Rightarrow x = y$ if ψ is injective). However this case is not very interesting since the aim is to use ψ to represent indistinguishability, as shown next. This requires ψ to be non-injective to allow for the possibility to have distinct elements considered as indistinguishable.

The similarity s and the pseudo-metric d_ψ are linked by the following relation: $\forall(x, y) \in \mathcal{T}^2 \setminus (0_\mathcal{T}, 0_\mathcal{T})$, $1 - s(x, y) = \frac{d_\psi(x,y)}{w(\psi(x)\vee\psi(y))}$ and $1 - s(0_\mathcal{T}, 0_\mathcal{T}) = d_\psi(0_\mathcal{T}, 0_\mathcal{T}) = 0$. The similarity is then a normalized version of d_ψ. If moreover $w \circ \psi$ satisfies the conditions of Corollary 1 or Proposition 1, then this normalized version is a pseudo-metric.

We also have the following additional properties (the proof is straightforward, hence omitted here).

Proposition 9. *Let w and ψ defined on $(\mathcal{T}, \preceq)$ as in Definition 3, and d and d_ψ as in Theorem 1 and Proposition 8:*

- *two elements of the lattice that are equivalent up to ψ have a zero distance: $\forall(x, y) \in \mathcal{T}^2$, $\psi(x) = \psi(y) \Rightarrow d_\psi(x, y) = 0$;*
- *if ψ is a morphological filter (i.e. increasing and idempotent), then $\forall(x, y) \in \mathcal{T}^2$, $x \preceq y \Rightarrow d_\psi(x, y) = w(\psi(y)) - w(\psi(x))$, and $d_{\psi\psi} = d_\psi$;*
- *if ψ is moreover anti-extensive (i.e. ψ is an opening), then $\forall x \in \mathcal{T}$, $d(x, \psi(x)) = w(x) - w(\psi(x))$. If ψ is extensive (i.e. ψ is a closing), then $\forall x \in \mathcal{T}$, $d(x, \psi(x)) = w(\psi(x)) - w(x)$;*
- *let us denote by $\mathrm{Inv}(\psi)$ the set of invariants by ψ (i.e. $x \in \mathrm{Inv}(\psi) \Leftrightarrow \psi(x) = x$). We have: $\forall(x, y) \in \mathrm{Inv}(\psi)^2$, $d_\psi(x, y) = d(x, y)$.*

The interest of the definitions and results of this section is that similarity and metrics are defined up to a transformation, which makes the results robust to variations of hypergraphs encoded by this transformation. The case where ψ is a filter is then of particular interest. For instance "noise" in the hypergraphs can be suppressed by an appropriate choice of ψ, and the original hypergraph and the filtered one will then be considered the same, with d_ψ being 0. For the comparison of two "noisy" hypergraphs, this distance allows ignoring the noise and focuses the comparison on the meaningful parts of the hypergraphs. Examples on simple hypergraphs will illustrate these effects in the next subsections.

4.3. Example for a dilation on $(\mathcal{P}(\mathcal{E}), \subseteq)$

Let us first consider the simple example introduced in [4]. For any hypergraph (V, E), we define a dilation δ on $(\mathcal{P}(\mathcal{E}), \subseteq)$ as:

$$\forall A \subseteq \mathcal{E}, \quad \delta(A) = \{e \in \mathcal{E} \mid v(A) \cap v(e) \neq \emptyset\}$$

where $v(A) = \cup_{e' \in A} v(e')$. Let $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ be two hypergraphs without empty hyperedge. We define a similarity function s by:

$$\forall A_1 \subseteq E_1,\ \forall A_2 \subseteq E_2, \quad s(A_1, A_2) = \frac{|\delta(A_1) \cap \delta(A_2)|}{|\delta(A_1) \cup \delta(A_2)|},$$

which corresponds to the similarity introduced in Definition 3 for $w = |.|$ and $\psi = \delta$.

Let us consider an example where hypergraphs are defined to represent image information. Vertices are pixels of the image, and hyperedges are subsets of pixels. Let us assume that the two images have the same support, and hence the corresponding hypergraphs have the same set of vertices. Let us denote them by $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$. In this example, the hyperedges were built from color and connectivity relations as follows: we define a neighborhood of each pixel x in each image i ($i = 1, 2$) as:

$$\Gamma^i_{\alpha,\beta}(x) = \{x' \mid d_C(\mathcal{I}_i(x), \mathcal{I}_i(x')) < \alpha \text{ and } d_N(x, x') \leq \beta\},$$

were d_C denotes a distance in the color space (or gray scale), $\mathcal{I}_i$ denotes the color or the intensity function in image i, d_N denotes the distance in the spatial domain and α and β are two parameters to tune the extent of the neighborhood (different values could be chosen for each image). The set of hyperedges E_i is then defined as the set of $\Gamma^i_{\alpha,\beta}(x)$ for all $x \in V$.

Fig. 4. An image and a modified version where a line has been introduced. The third image illustrates the dissimilarity (darkest gray levels). The global similarity is 0.96. The last image shows the inverted difference between the images (computed pixel-wise).

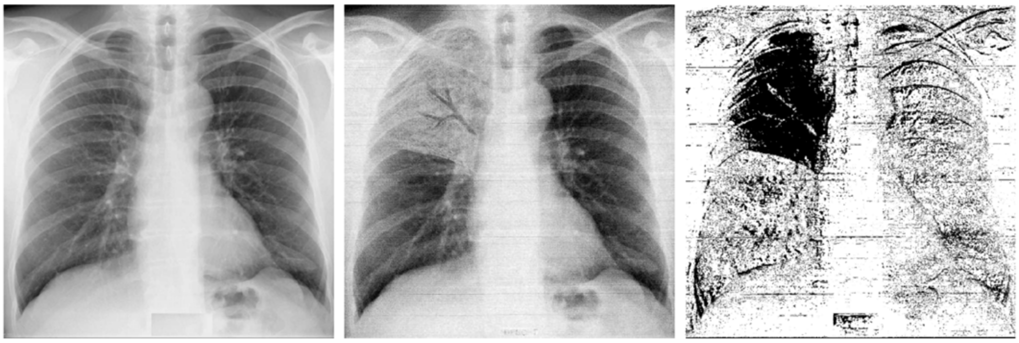

Fig. 5. Similarity (right) between normal (left) and pathological (center) lungs.

A weighted average similarity can be defined as:

$$s(H_1, H_2) = \frac{1}{2}\left(\frac{1}{\sum\limits_{e\in E_1} |\delta(e)|}\sum_{e\in E_1} s(e, E_2)|\delta(e)| + \frac{1}{\sum\limits_{e'\in E_2} |\delta(e')|}\sum_{e'\in E_2} s(e', E_1)|\delta(e')|\right)$$

where

$$s(e, E_2) = \max_{e'\in E_2} \frac{|\delta(e) \cap \delta(e')|}{|\delta(e) \cup \delta(e')|}$$

and a similar expression for $s(e', E_1)$.

In the following figures, the similarity image is displayed by representing as a gray level the value of $\frac{1}{2}(s(e(x), E_2) + s(e'(x), E_1))$ at each pixel x (0 being white and 1 being black), where $e(x) = \Gamma^1_{\alpha,\beta}(x)$ is computed for the first image and $e'(x) = \Gamma^2_{\alpha,\beta}(x)$ is computed for the second image.

In the example in Fig. 4, the similarity between the left image and its modification with an additional line is equal to 0.96. The figure on the right illustrates the dissimilarity between the two images. The dilation leads to more robustness to small and non relevant variations in the images (without the dilation, the similarity would be 0.94). The direct pixel-wise difference between the two images does not show as clearly the modifications. In this very simple case, neighbors of the additional line have non-zero values because of the dilation. This effect would be even more interesting in cases of less abrupt changes in the image. Similarly, in the example in Fig. 5 the similarity is computed between registered X-ray images of normal and pathological lungs, highlighting the pathological region. Its value is 0.75 (and 0.61 without dilation).

Another example is illustrated in Fig. 6, where two images exhibiting some differences are compared. The comparison is illustrated in four sub-images. The similarity is equal to 1 in the top left part, to 0.75 in the top right part, to 0.98 in the bottom left part and to 0.97 in the bottom right part. Again this fits what could be intuitively expected. The global similarity, computed over the whole images, is equal to 0.93. The subdivision (even very simple here) allows us to better localize the differences. To illustrate further the potential interest of the proposed approach, the pixel-wise difference between the two images is displayed in Fig. 6(d). It clearly shows that it is much more sensitive to small local variations, leading to a more noisy result, and the differences between the two images do not appear as clearly as with the proposed approach (similarity values are higher). Fig. 6(e) illustrates a change of contrast in image (b). The similarity using the proposed approach and the pixel-wise difference are displayed in (f) and (g), respectively. Note for instance the non-zero constant large region in (g), revealing the sensitivity to contrast change of a simple difference. Similar experiences with other types of modification or addition of noise could be done in future work. In case of noise, instead of using dilations, using morphological filters might be better suited.

This example is a preliminary step towards applications of the proposed approach for evaluating differences between images, which could be extended to change detection. However more experiments are needed before comparing with existing change detection methods. This is left for future work.

4.4. Example for an opening on $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$

Let us now consider $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ and $\mathcal{T}_2 = (\mathcal{P}(\mathcal{E}), \subseteq)$. As in [4], we define a dilation from $\mathcal{T}_2$ into $\mathcal{T}_1$ as:

$$\begin{aligned}\forall e \in E, \quad B_e &= \delta(\{e\}) = \{x \in \mathcal{V} \mid \exists e' \in \mathcal{E}, x \in v(e') \text{ and } v(e) \cap v(e') \neq \emptyset\} \\ &= \cup\{v(e') \mid v(e') \cap v(e) \neq \emptyset\},\end{aligned}$$

and the dilation of any subset of $\mathcal{E}$ is defined using the sup-generating property. Since we assumed that $v(e)$ is always non-empty, then we have $\forall e \in E$, $v(e) \subseteq B_e$.

Note that such dilations from the vertex set to the edge or hyperedge set have been introduced previously on graphs [10,11], and particular hypergraphs such as the ones associated with the hexagonal grid [21] or simplicial complexes [12].

Proposition 10. *The adjoint erosion ε of δ, from $\mathcal{T}_1$ into $\mathcal{T}_2$ is given by:*

$$\begin{aligned}\forall V \in \mathcal{P}(\mathcal{V}), \quad \varepsilon(V) &= \cup\{E \in \mathcal{P}(\mathcal{E}) \mid \forall e \in E,\ \delta(\{e\}) \subseteq V\} \\ &= \{e \in \mathcal{E} \mid \forall e' \in \mathcal{E},\ v(e') \cap v(e) \neq \emptyset \Rightarrow v(e') \subseteq V\}.\end{aligned}$$

The opening $\gamma = \delta\varepsilon$ is defined from $\mathcal{T}_1$ into $\mathcal{T}_1$ and is expressed as:

$$\begin{aligned}\forall V \in \mathcal{P}(\mathcal{V}), \quad \gamma(V) &= \cup\{v(e') \mid \exists e \in \varepsilon(V), v(e') \cap v(e) \neq \emptyset\} \\ &= \cup\{B_e \mid B_e \subseteq V\}.\end{aligned}$$

Proof. In any lattices $(\mathcal{T}, \preceq)$, $(\mathcal{T}', \preceq')$ the adjoint erosion of a dilation from $\mathcal{T}'$ into $\mathcal{T}$ is an operator from $\mathcal{T}$ into $\mathcal{T}'$ expressed $\forall x \in \mathcal{T},\ \varepsilon(x) = \bigvee\{y \in \mathcal{T}',\ \delta(y) \preceq x\}$ [15], hence the expression of the adjoint erosion. The form of the opening is also derived from a general property of openings defined as the composition of morphological erosion and dilation, expressing the opening as the supremum (union here) of all structuring elements included in V [15,26]. ■

The result of the opening is the set of vertices of the hyperedges whose neighbors (as defined by B_e) are in V and vertices of these neighbors. The example in Fig. 7 illustrates that vertices that belong to "incomplete" hyperedges (i.e. for which the set of vertices is not completely included in V) are removed. Note that vertices of complete hyperedges could be removed too.

When computing the similarity $s(V, V') = \frac{|\gamma(V)\cap\gamma(V')|}{|\gamma(V)\cup\gamma(V')|}$, it is clear that if V' differs from V only by vertices from incomplete hyperedges, then $s(V, V') = 1$. The similarity is then robust to noise vertices. In particular $s(V, \gamma(V)) = 1$ since $\gamma(\gamma(V)) = V$. Other examples are shown in Fig. 8, which have the same opening as in Fig. 7 (right). Hence all these subsets of vertices have a similarity equal to 1 (i.e. they are equivalent up to γ and only differ by their isolated vertices).

Let us now consider another example of opening.

Proposition 11. *The operator γ' from $\mathcal{T}_1$ into $\mathcal{T}_1$ defined as $\forall V \in \mathcal{P}(\mathcal{V}),\ \gamma'(V) = \cup_{e\in\mathcal{E}}\{v(e) \mid v(e) \subseteq V\}$ is an opening. It can also be expressed as $\gamma' = \delta'\varepsilon'$, where δ' is the dilation from $\mathcal{T}_2$ into $\mathcal{T}_1$ defined as $\forall E \in \mathcal{P}(\mathcal{E}),\ \delta'(E) = \cup_{e\in E} v(e)$ and ε' is the adjoint erosion.*

Proof.
- $\forall V \in \mathcal{P}(\mathcal{V}), \forall V' \in \mathcal{P}(\mathcal{V}),\ V \subseteq V' \Rightarrow (\forall e \in \mathcal{E}, v(e) \subseteq V \Rightarrow v(e) \subseteq V')$, and $\gamma'(V) \subseteq \gamma'(V')$. Therefore γ' is increasing.
- By definition $\forall V \in \mathcal{P}(\mathcal{V}),\ \gamma'(V) \subseteq V$ and γ' is anti-extensive.
- Since $\forall V \in \mathcal{P}(\mathcal{V}),\ \gamma'(V)$ only contains vertices of complete hyperedges (i.e. is the union of some $v(e)$), it is invariant by γ', i.e. $\gamma'(\gamma'(V)) = \gamma'(V)$, and γ' is therefore idempotent.
- The adjoint erosion of δ' is defined from $\mathcal{T}_1$ into $\mathcal{T}_2$ and computed as:

  $$\forall V \in \mathcal{P}(\mathcal{V}), \quad \varepsilon'(V) = \cup\{E \in \mathcal{P}(\mathcal{E}) \mid \delta'(E) \subseteq V\} = \{e \in \mathcal{E} \mid v(e) \subseteq V\}$$

  which is the set of complete hyperedges formed by vertices of V. Then $\delta'\varepsilon'(V)$ is the set of vertices of these complete hyperedges, and is therefore equal to $\gamma'(V)$. ■

This opening keeps all vertices of complete hyperedges, i.e. the ones that are "well connected" in the hypergraph. This can be used for filtering hypergraphs by keeping only vertices of complete hyperedges, which can be interesting for indexing and retrieval purposes (vertices from incomplete hyperedges being then considered as noise). Invariants of γ' are the subsets V that contain only vertices of complete hyperedges. An example is illustrated in Fig. 9. The subset V is shown in red on the left and its opening $\gamma'(V) = v(e_5)$ in blue on the right. Note that for this example we have $\gamma(V) = \emptyset$, thus illustrating the difference between γ and γ'. More generally, we always have $\gamma(V) \subseteq \gamma'(V)$.

Again this makes the similarity robust to vertices which belong to incomplete hyperedges. We have $s(V, V') = 0$ iff $V \cap V'$ is the set of noise vertices.

If we consider a binary version of the similarity, i.e. V and V' are equivalent iff $\gamma'(V) = \gamma'(V')$, then equivalence classes are built of subsets of $\mathcal{V}$ which contain the vertices of the same complete hyperedges. In particular V and $\gamma'(V)$ belong to the

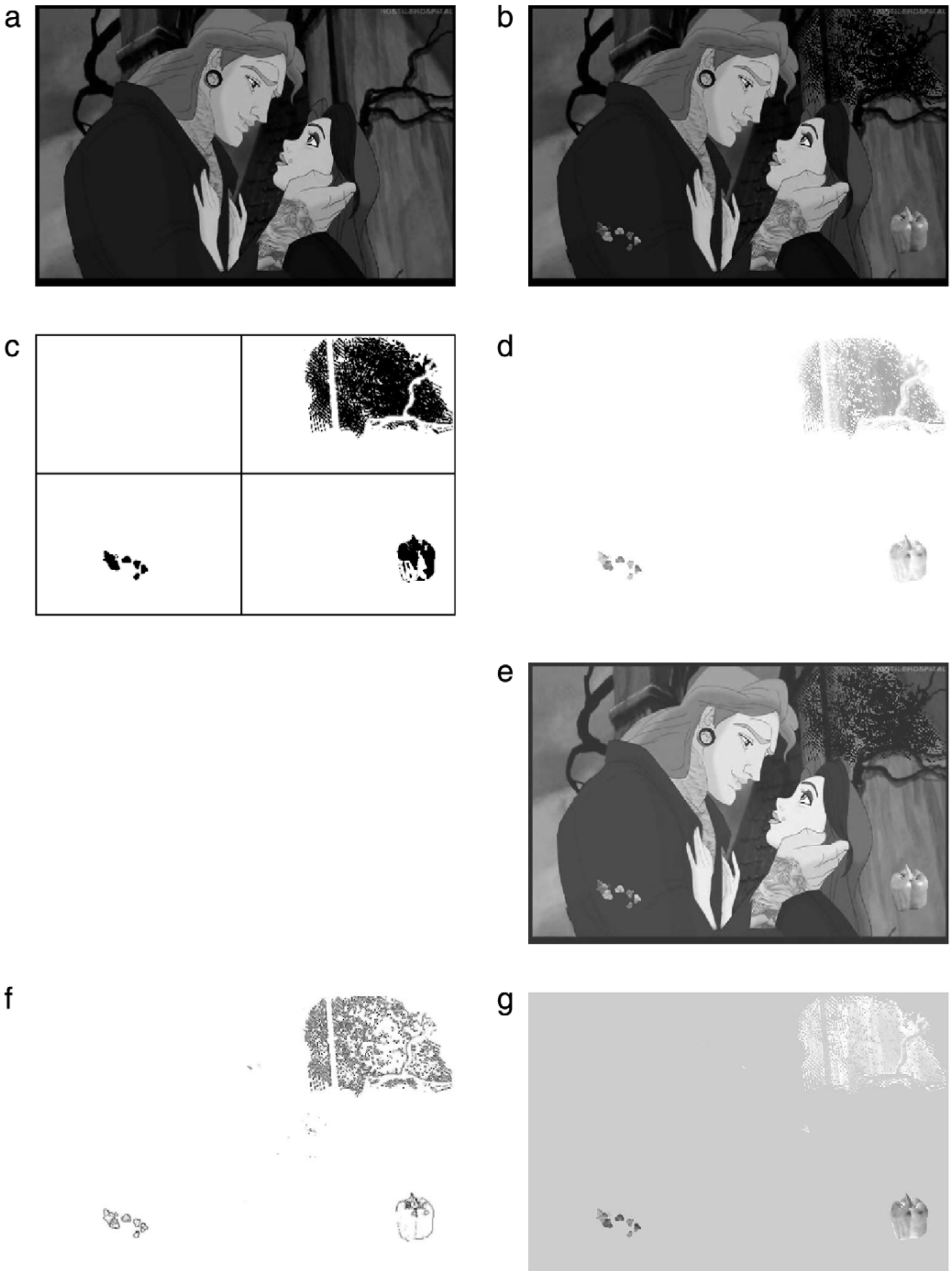

Fig. 6. (a), (b) Two images with some differences. (c) Dissimilarity image using the proposed approach. (d) Pixel-wise difference between the two images (with inverted contrast for interpretation as a similarity). (e) Change of contrast in image (b). (f) Dissimilarity image between (a) and (e). (g) Pixel-wise difference between (a) and (e).

same equivalence class. Using this equivalence relation can be useful for robust indexing and retrieval, for robust entropy definition, etc.

4.5. Example on $\mathcal{T}_3 = (\{H_j, j \in J\}, \preceq)$

Let us now consider the most interesting case where $\mathcal{T}$ is the lattice of hypergraphs $\mathcal{T}_3 = (\{H_j, j \in J\}, \preceq)$. We again consider opening, as an example of morphological filter, to illustrate our purpose. Similar examples can be found with other types of filters.

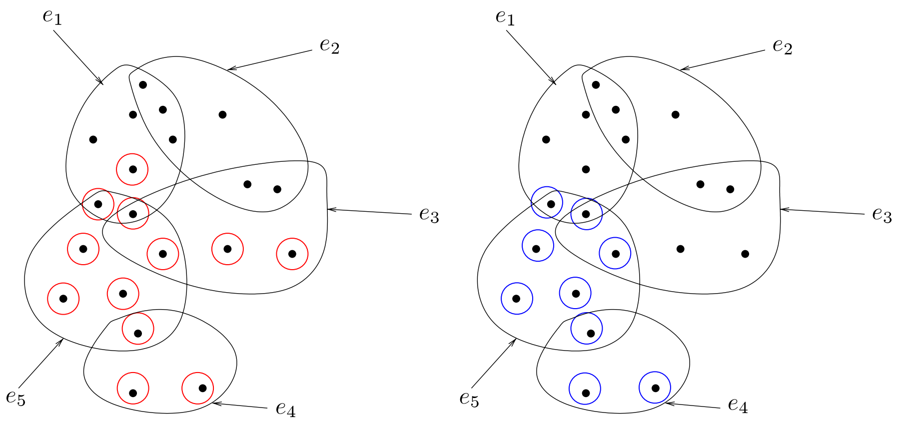

Fig. 7. Example of an opening from $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ into $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ [4]. The red circled vertices on the left represent V. Its opening is shown in blue on the right, and is equal to $\gamma(V) = \delta(e_4) = v(e_4) \cup v(e_5)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

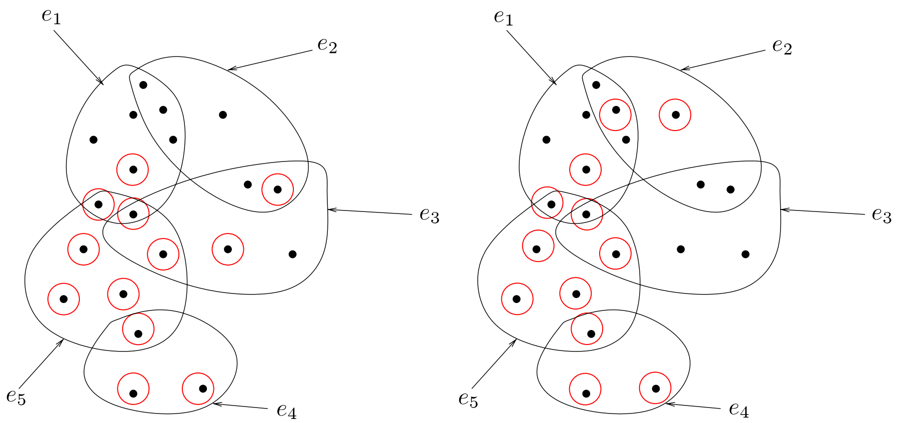

Fig. 8. Two other subsets of $\mathcal{V}$ having the same opening (shown in blue on the right in Fig. 7), i.e. vertices of e_4 and e_5. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

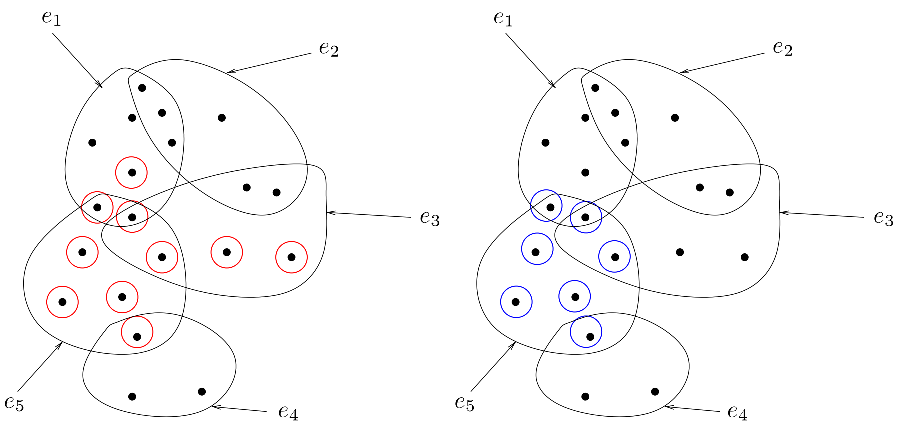

Fig. 9. Subset V (in red) and its opening $\gamma'(V)$ (in blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proposition 12. *The operator γ_1 defined for each hypergraph $H = (V, E)$ in $\mathcal{T}_3$ by:*

$$\gamma_1(H) = (\cup_{e \in E}\, v(e), E) = (V \setminus V_{\setminus E}, E),$$

i.e. the operator that removes the isolated vertices, is an opening.

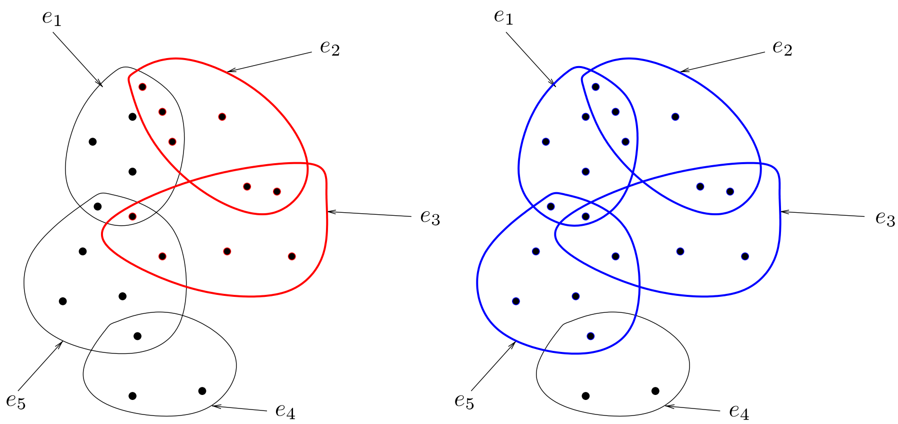

Fig. 10. The figure on the left represents $\mathcal{V}$ (vertices represented as points) and $\mathcal{E}$ (hyperedges represented as closed lines). The red lines indicate the hyperedges of H. The vertices of H are the points enclosed in these lines. The blue lines on the right represent the hyperedges of $\delta(H)$ and its vertices are the points enclosed in these lines. For this example, $\varepsilon(H)$ and $\gamma_2(H)$ are empty. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. Extensivity and increasingness are straightforward. The idempotence follows from the fact that γ_1 removes isolated vertices from the vertex set. Hence the result is invariant by a further application of the operation. ■

Let us now consider the dilation introduced in [4] on this lattice. The canonical decomposition of H, from its sup generating property, is expressed as:

$$H = (\vee_{e\in E}(v(e), \{e\})) \vee (\vee_{x\in V_{\setminus E}}(\{x\}, \emptyset)).$$

From this decomposition, a dilation is defined as:

$$\forall x \in V_{\setminus E}, \quad \delta(\{x\}, \emptyset) = (\{x\}, \emptyset),$$

for isolated vertices, and for elementary hypergraphs associated with hyperedges:

$$\forall e \in E, \quad \delta(v(e), \{e\}) = (\cup\{v(e') \mid v(e') \cap v(e) \neq \emptyset\},\ \{e' \in \mathcal{E} \mid v(e') \cap v(e) \neq \emptyset\}).$$

The dilation of any H is then derived from its decomposition and from the commutativity of dilation with the supremum.

In the particular case where H has no isolated vertices, then it is sufficient to consider the hyperedges (since the set of vertices is automatically equal to $\cup_{e\in E} v(e)$), and δ can be written in a simpler form as:

$$\delta(\{e\}) = B_e = \{e' \in \mathcal{E} \mid v(e) \cap v(e') \neq \emptyset\},$$

and

$$\delta(E) = \cup_{e\in E}\, \delta(\{e\}) = \{e' \in E \mid \exists e \in E, v(e') \cap v(e) \neq \emptyset\}.$$

An example is illustrated in Fig. 10.

Proposition 13. *Let us consider hypergraphs without isolated vertices. The adjoint erosion of δ is given by:*

$$\forall E \in \mathcal{P}(\mathcal{E}), \quad \varepsilon(E) = \cup\{E' \in \mathcal{P}(\mathcal{E}) \mid \delta(E') \subseteq E\} = \{e \in \mathcal{E} \mid B_e \subseteq E\}$$

and

$$\varepsilon(H) = (\cup_{e\in\varepsilon(E)}\, v(e), \varepsilon(E)).$$

The opening $\gamma_2 = \delta\varepsilon$ is then:

$$\gamma_2(E) = \cup_{B_e\subseteq E}\, B_e,$$

and

$$\gamma_2(H) = (\cup_{e\in\gamma_2(E)}\, v(e), \gamma_2(E)).$$

Proof. This result is similar to Proposition 10 on the lattices built on vertices. ■

In Fig. 10, the erosion of $H = (v(e_2) \cup v(e_3), \{e_2, e_3\})$ shown in red is empty, and the opening is empty as well.

In Fig. 11, the erosion of $H = (\cup_{i\in\{1,2,3,5\}}\, v(e_i), \{e_1, e_2, e_3, e_5\})$ is equal to $(\cup_{i=1}^{3}\, v(e_i), \{e_1, e_2, e_3\})$ (it does not contain e_5 since this hyperedge overlaps with e_4 which is not in H) and the opening is $\gamma_2(H) = H$.

Another example is illustrated in Fig. 12, showing the filtering effect of this opening.

If we now consider the more general case where the hypergraphs have isolated vertices, since these are preserved by dilation, they are also preserved in the adjoint erosion and in the derived opening. These isolated vertices do not induce any

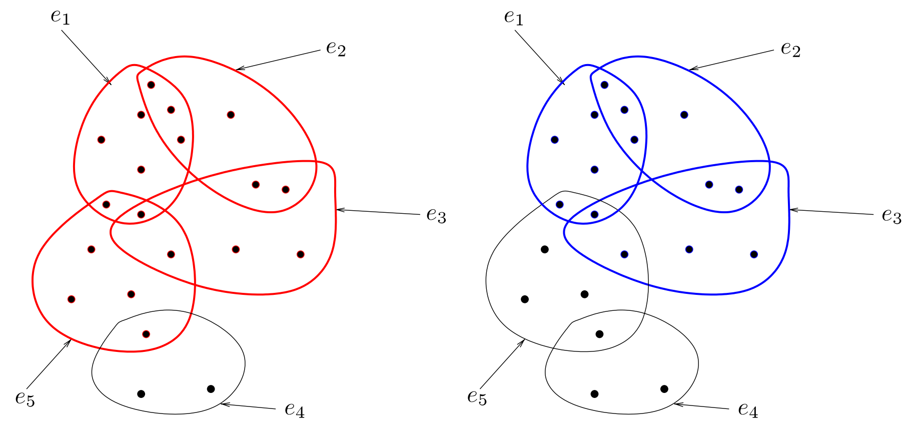

Fig. 11. H is represented on the left using the same conventions as in Fig. 10 (i.e. $\mathcal{V}$ is the set of all points and $\mathcal{E}$ the set of all closed lines, and H is represented in red). Its erosion is shown on the right (in blue) and $\gamma_2(H) = H$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

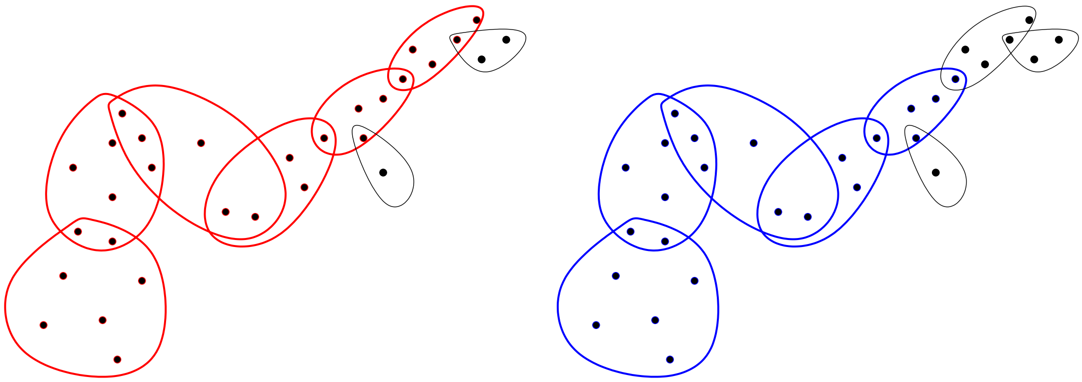

Fig. 12. H is represented on the left using the same conventions as in Fig. 10. Its opening $\gamma_2(H)$ is shown on the right.

change when using γ_2 or not for computing the similarity or the distance. For instance if H' is equal to H plus k additional isolated vertices, then $d(H, H') = d_{\gamma_2}(H, H') = k$.

Let us now consider the height as a valuation on $\mathcal{T}_3$. As shown in Proposition 6, we have $\forall H = (V, E) \in \mathcal{T}_3$, $w(H) = |V| + |E|$. We have: $d(H, \gamma_1(H)) = |V_{\setminus E}|$, which is the number of isolated vertices in H (the distance evaluates the amount of "noise" in H if isolated vertices are interpreted as noise vertices). If H and H' differ only by isolated vertices, then $d_{\gamma_1}(H, H') = 0$. If we consider now γ_2, then the general results expressed in Proposition 9 hold, along with the associated interpretation.

Let us give a few simple examples:

- in Fig. 13, $H = (\{v_1, v_2\} \cup v(e_2) \cup v(e_3), \{e_2, e_3\})$ shown on the left contains two isolated vertices. Its opening $\gamma_1(H) = (v(e_2) \cup v(e_3), \{e_2, e_3\})$ is displayed on the right (the isolated vertices are suppressed), and we have $d(H, \gamma_1(H)) = |V_{\setminus E}| = 2$;
- for H depicted in Fig. 11, $d(H, \gamma_2(H)) = d(H, H) = 0$;
- for H depicted in Fig. 12, $d(H, \gamma_2(H)) = 4 + 1 = 5$;
- two hypergraphs H_1 and H_2 having the same opening by γ_2 are displayed in Fig. 14. Hence $d_{\gamma_2}(H_1, H_2) = 0$. Now if k isolated vertices are added to one of the two hypergraphs, their opening will stay the same up to these isolated vertices, and $d_{\gamma_2}(H_1, H_2) = k$.

Closings can be built in a similar way as openings. In a dual way to the construction of γ_1 and γ_2, we propose the following operators, one which "completes" all incomplete hyperedges, and the second one which is built from erosion and dilation.

Proposition 14. *The operator φ_1 defined for each hypergraph $H = (V, E)$ in $\mathcal{T}_3$ by:*

$$\varphi_1(E) = \{e \in \mathcal{E} \mid \exists x \in V, x \in v(e)\} = \{e \in \mathcal{E} \mid v(e) \neq \emptyset\},$$
$$\varphi_1(H) = (\cup_{e \in \varphi_1(E)} v(e), \varphi_1(E))$$

is a closing.

(Since no confusion can arise, we use the same notation for the operator applied on hyperedges and on the whole hypergraph.)

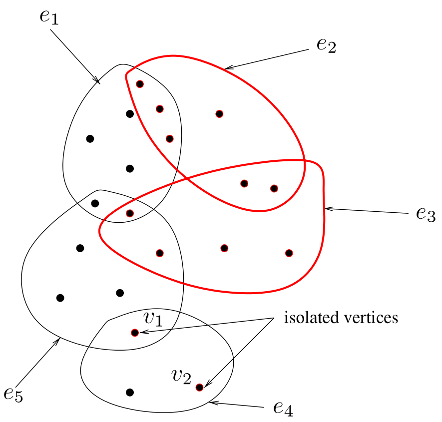

Fig. 13. A hypergraph $H = (\{v_1, v_2\} \cup v(e_2) \cup v(e_3), \{e_2, e_3\})$ containing two isolated vertices v_1 and v_2 (left) and its opening $\gamma_1(H) = (v(e_2) \cup v(e_3), \{e_2, e_3\})$ (right).

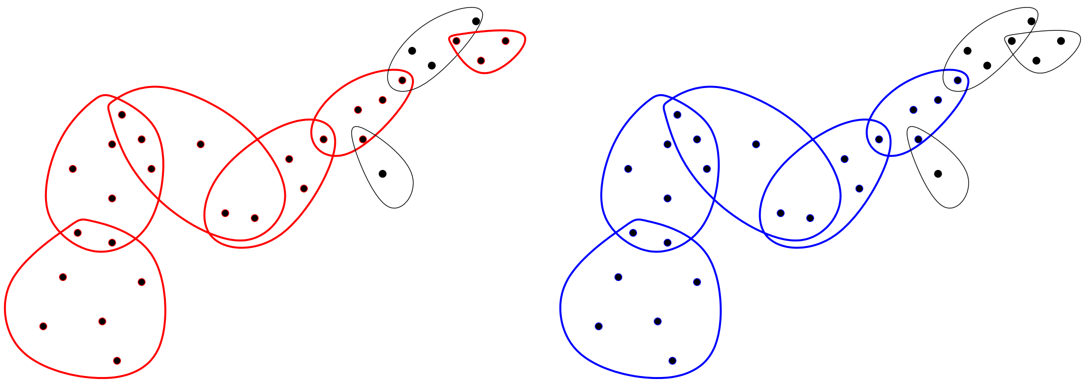

Fig. 14. Two hypergraphs H_1 (left) and H_2 (right). Their openings are $\gamma_2(H_1) = \gamma_2(H_2) = H_2$ and $d_{\gamma_2}(H_1, H_2) = 0$.

Proposition 15. *Let us consider δ and ε as in Proposition 13 and their extension to the cases where hypergraphs can have isolated vertices. Then $\varphi_2 = \varepsilon\delta$ is a closing.*

Proofs are similar to the ones for openings.
An example is illustrated in Fig. 15.

5. Conclusion

The contribution of this paper is an original framework to define similarities between hypergraphs, a problem that was not much addressed before. It builds on the lattice structure of hypergraphs, on the notion of valuation in complete lattices, and on mathematical morphology. Beside some general results on similarities, the proposed framework offers new tools for defining similarity measures and pseudo-metrics, which are robust to variations (encoded by morphological operators) of hypergraphs.

They can be incorporated in existing systems for hypergraph-based feature selection, indexing, retrieval, matching. As an example, let us consider the equivalence relation on any lattice of hypergraphs $\mathcal{T}$ defined by $\forall(x, y) \in \mathcal{T}^2, xRy \Leftrightarrow \psi(x) = \psi(y)$ where ψ is a morphological operator on $\mathcal{T}$. This equivalence relation induces a partition of $\mathcal{T}$, denoted by $\mathcal{T} = \cup_i T_i$. A discrete probability distribution can then be defined as $p_i = \frac{|T_i|}{|\mathcal{T}|}$ from which an entropy of $\mathcal{T}$ (up to ψ) can be derived. This defines a new entropic criterion that can be used in feature selection methods such as [31].

Future work aims at exploring other examples of morphological operators in the proposed framework (for instance as the ones defined in [12] on simplicial complexes, such as alternate sequential filters, or auto-dual filters), and weaker forms of valuations, by considering the sub- or supra-modular cases [27]. Applications to images could also be the scope of future research, including the choice of the appropriate morphological operator to achieve robustness with respect to change of contrast, noise, etc., and applications to indexing, matching, retrieval.

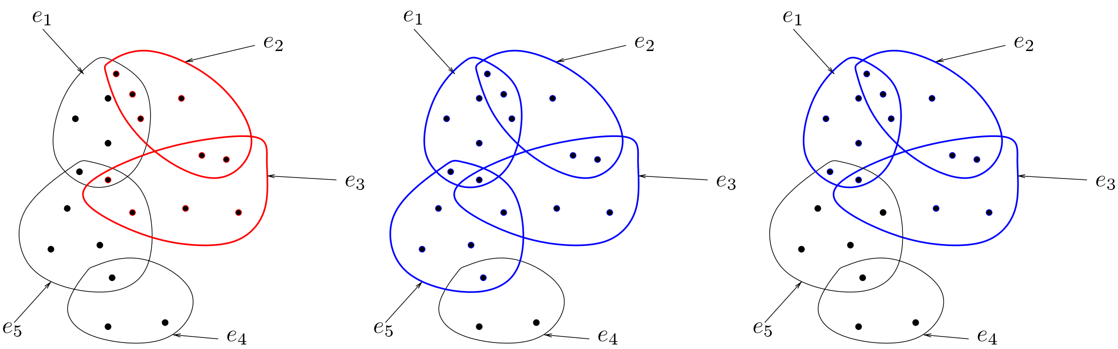

Fig. 15. The figure on the left represents $\mathcal{V}$ (vertices represented as points) and $\mathcal{E}$ (hyperedges represented as closed lines). The red lines indicate the hyperedges of H. The vertices of H are the points enclosed in these lines. The blue lines in the middle represent the hyperedges of $\varphi_1(H)$ and its vertices are the points enclosed in these lines. Similarly, the blue lines on the right represent the hyperedges if $\varphi_2(H)$ and its vertices are the points enclosed in these lines. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

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