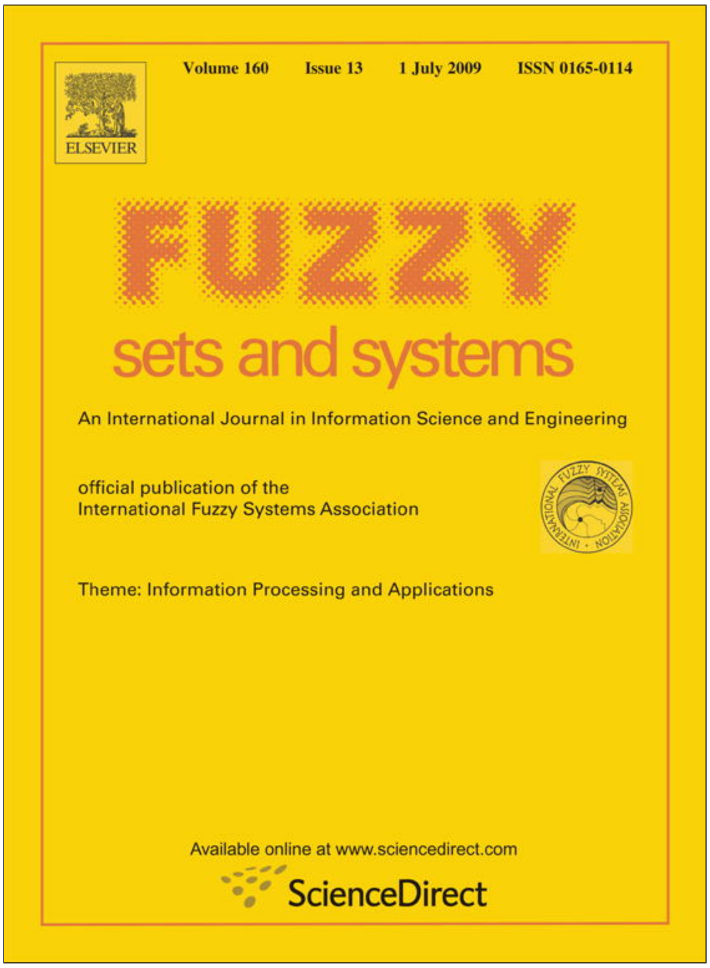
Volume 160 Issue 13 1 July 2009 ISSN 0165-0114
ELSEVIER
FUZZY
sets and systems
An International Journal in Information Science and Engineering
official publication of the
International Fuzzy Systems Association
Theme: Information Processing and Applications
Available online at www.sciencedirect.com
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ScienceDirect
ELSEVIER
Fuzzy Sets and Systems 160 (2009) 1858 – 1867
FUZZY sets and systems
www.elsevier.com/locate/fss

Duality vs. adjunction for fuzzy mathematical morphology and general form of fuzzy erosions and dilations

Isabelle Bloch*

TELECOM ParisTech (ENST), Signal and Image Processing Department, CNRS UMR 5141 LTCI, 46 rue Barrault, 75013 Paris, France

Received 13 August 2007; received in revised form 10 August 2008; accepted 16 January 2009
Available online 3 February 2009

Abstract

We establish in this paper the link between the two main approaches for fuzzy mathematical morphology, based on duality with respect to complementation and on the adjunction property, respectively. We also prove that the corresponding definitions of fuzzy dilation and erosion are the most general ones if a set of classical properties is required.

Keywords: Fuzzy mathematical morphology; Adjunction; Duality; Erosion; Dilation

1. Introduction

Extending mathematical morphology to fuzzy sets was addressed by several authors during the last years. Some definitions just consider grey levels as membership functions, or use binary structuring elements. Here we restrict ourselves to really fuzzy approaches, where fuzzy sets have to be transformed according to fuzzy structuring elements. Initial developments can be found in the definition of fuzzy Minkowski addition [13]. Then this problem has been addressed by several authors independently, e.g. [1,2,5,9,8,11,21,22,24,31]. These works can be divided into two main approaches. In the first one [5,9,10], an important property that is put to the fore is the duality between erosion and dilation. A second type of approach is based on the notions of adjunction and fuzzy implication, and was formalized in [11]. The aim of this paper is twofold. First, we will clarify the links between both approaches (which are summarized in Section 3) and establish the conditions of their equivalence, as well as the class of operators satisfying these conditions (Section 4). Then, in Section 5, we will prove that the definitions of dilation and erosion in these approaches are the most general ones if we want them to share a set of classical properties with standard mathematical morphology. The main contributions of this paper consist of the technical results of Sections 4 and 5, extending a preliminary version [4].

The motivation for this work and the interest of linking duality and adjunction approaches are first to clarify the status of these approaches and to gather results that are somewhat spread in the literature, and, more importantly, expressed from different points of view. Although there are obvious links and similarities between algebraic, set theoretical, logical, and morphological perspectives, each of them brings an interesting insight and leads to results that can be used

* Tel.: +33 1 45 81 75 85; fax: +33 1 45 81 37 94.
E-mail address: Isabelle.Bloch@enst.fr.

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doi:10.1016/j.fss.2009.01.006

from another perspective. Here we address the question of duality and adjunction mainly under the light of mathematical morphology in a set theoretical setting, bearing in mind applications in image processing where objects in images are considered as sets or fuzzy sets, to be studied based on their shape and their relations to their neighborhood. This point of view provides a new perspective on partially existing results. Duality with respect to complementation, which was advocated in the first developments of mathematical morphology [29], is then important to handle in an consistent way an object and the background for many applications. Therefore it is useful to know exactly under which conditions this property may hold, so as to choose the appropriate operators if it is needed for a specific problem. On the other hand, adjunction is a major feature of the "modern" view of mathematical morphology, with strong algebraic bases in the framework of complete lattices [25]. This framework is now widely considered as the most interesting one, and extending mathematical morphology to fuzzy sets in this framework inherits a set of powerful and important properties.

2. Basic notions and notations

For the sake of completeness, we recall in this section a few definitions and introduce the notations used in the paper.

A fuzzy implication I is a mapping from $[0,1]\times[0,1]$ into $[0,1]$ which is decreasing in the first argument, increasing in the second one and satisfies $I(0,0)=I(0,1)=I(1,1)=1$ and $I(1,0)=0$.

A fuzzy conjunction C is a mapping from $[0,1]\times[0,1]$ into $[0,1]$ which is increasing in both arguments and satisfies $C(0,0)=C(1,0)=C(0,1)=0$ and $C(1,1)=1$. If C is also associative and commutative and satisfies $\forall\alpha\in[0,1], C(\alpha,1)=C(1,\alpha)=\alpha$, it is a t-norm, and will then be denoted by T in the following.

A fuzzy disjunction S is a mapping from $[0,1]\times[0,1]$ into $[0,1]$ which is increasing in both arguments and satisfies $S(1,1)=S(1,0)=S(0,1)=1$ and $S(0,0)=0$. If S is also associative and commutative and satisfies $\forall\alpha\in[0,1],\ S(\alpha,0)=S(0,\alpha)=\alpha$, it is a t-conorm.

A negation (or complementation in a set theoretical terminology) is a mapping c from $[0,1]$ into $[0,1]$, which is decreasing and satisfies $c(1)=0$ and $c(0)=1$. In this paper, we consider only involutive negations, i.e. such that $\forall\alpha\in[0,1],\ c(c(\alpha))=\alpha$, the interest of which being highlighted e.g. in [19].

Typical examples are minimum, product, Lukasiewicz operators for t-norms, maximum, algebraic sum, Lukasiewicz operators for t-conorms, $c(\alpha)=1-\alpha$ for negation. Other examples will be considered later, in Section 4.

These connectives can be linked to each other in different ways. In particular, using duality and residuation principles. The following examples will be used in this paper:

- a t-conorm can be derived from a t-norm and a negation as: $\forall(\alpha,\beta)\in[0,1]^2,\ S(\alpha,\beta)=c(T(c(\alpha),c(\beta)))$;
- an implication I induces a negation c defined as: $\forall\alpha\in[0,1],\ c(\alpha)=I(\alpha,0)$;
- an implication I can be defined from a negation c and a conjunction C by: $\forall(\alpha,\beta)\in[0,1]^2,\ I(\alpha,\beta)=c(C(\alpha,c(\beta)))$;
- conversely, a conjunction C can be defined from a negation c and an implication I: $\forall(\alpha,\beta)\in[0,1]^2,\ C(\alpha,\beta)=c(I(\alpha,c(\beta)))$;
- a common way to define an implication I (called S-implication), consists in deriving it from a t-conorm S and a negation c [12]:

$$\forall(\alpha,\beta)\in[0,1]^2,\quad I(\alpha,\beta)=S(c(\alpha),\beta),\tag{1}$$

  this equation translating directly the crisp logical equivalence between $(\varphi\Rightarrow\psi)$ and $(\psi\vee\neg\varphi)$;
- another interesting and usual approach to derive an implication I (called R-implication) is to apply the residuation principle, from a t-norm T:

$$\forall(\alpha,\beta)\in[0,1]^2,\quad I(\alpha,\beta)=\sup\{\gamma\in[0,1],T(\alpha,\gamma)\leqslant\beta\}.\tag{2}$$

  This definition coincides with the previous one for particular forms of T, typically the Lukasiewicz t-norm.

Let $\mathcal{A}$ be a mapping from B to A and $\mathcal{B}$ be a mapping from A to B. The pair $(\mathcal{A},\mathcal{B})$ is a Galois connection if:

$$\forall a\in A,\ \forall b\in B,\quad a\leqslant\mathcal{A}(b)\ \Leftrightarrow\ b\leqslant\mathcal{B}(a).$$

In this definition, $\mathcal{A}$ and $\mathcal{B}$ play symmetrical roles.

Let δ be a mapping from A to B and ε be a mapping from B to A. The (ordered) pair (ε,δ) is an adjunction if:

$$\forall a\in A,\ \forall b\in B,\quad \delta(a)\leqslant b\ \Leftrightarrow\ a\leqslant\varepsilon(b).$$

Note that in comparison to the Galois connection, the ordering on B is reversed ($\delta(a) \leqslant b$ instead of $b \leqslant \delta(a)$) and ε and δ do not play symmetrical roles. This distinction between Galois connection and adjunction is usual in the mathematical morphology community, while an adjunction is sometimes also called Galois connection in some domains (e.g. in [16,20]). Here we will use the adjunction terminology.

Let us now consider the fuzzy connectives. A pair of operators (I, C) are said adjoint if:

$$\forall(\alpha, \beta, \gamma) \in [0, 1]^3, \quad C(\alpha, \beta) \leqslant \gamma \;\Leftrightarrow\; \beta \leqslant I(\alpha, \gamma). \tag{3}$$

The adjoint of a conjunction is a residual implication (there is an equivalence between the residuation principle and the adjunction principle, resulting from general properties of adjunctions, see e.g. [16,17]). This explains why the adjunction property is sometimes called residuation principle (e.g. [15]). Moreover, this property, expressed for a t-norm, is equivalent to the left-continuity of the t-norm [15,20].

As shown in [14], both types of implication are equivalent if and only if the t-norm satisfies $T(\alpha, \beta) \leqslant \gamma \Leftrightarrow T(\alpha, c(\gamma)) \leqslant c(\beta)$, i.e. the adjunction property, by identifying $T(\alpha, c(\gamma))$ and $c(I(\alpha, \gamma))$ for dual operators (see also [18,20], with interesting geometric applications). This property of T is also called rotation-invariance.

3. Summary of the two main approaches

Let us first briefly recall the two main approaches towards fuzzy mathematical morphology.

Fuzzy sets are defined on a space $\mathcal{S}$, through their membership functions from $\mathcal{S}$ into [0, 1]. The set of fuzzy sets on $\mathcal{S}$ is denoted by $\mathcal{F}$, and $\leqslant$ is the partial ordering defined by $\mu \leqslant \nu \Leftrightarrow \forall x \in \mathcal{S},\ \mu(x) \leqslant \nu(x)$. This defines a lattice $(\mathcal{F}, \leqslant)$.

A mapping δ on this lattice is a dilation if it commutes with the supremum. A mapping ε is an erosion if it commutes with the infimum.

We assume that $\mathcal{S}$ is endowed with an affine structure, and we consider morphological operations that are invariant by translation, and defined using structuring elements.

3.1. Fuzzy morphology by formal translation of crisp equations using t-norms and t-conorms

The first attempts to build fuzzy mathematical morphology were based on translating binary equations into fuzzy ones, as developed in [2,5]. This translation is done term by term, by substituting all crisp expressions by their fuzzy equivalents. For instance, intersection is replaced by a t-norm, union by a t-conorm, sets by fuzzy set membership functions, etc. This allows expressing erosion as a degree of inclusion and dilation as a degree of intersection.

An important property that was put to the fore in this approach is the duality between erosion and dilation.

In the crisp case, let $\varepsilon_B(X)$ denote the erosion of the set X (subset of $\mathcal{S}$) by the structuring element B (subset of $\mathcal{S}$), defined by

$$x \in \varepsilon_B(X) \;\Leftrightarrow\; B_x \subseteq X,$$

where B_x denotes B translated at point x. The translation of this expression into fuzzy terms leads to a natural way to define the erosion of a fuzzy set μ by a fuzzy structuring element ν, as

$$\forall x \in \mathcal{S}, \quad \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} S[c(\nu(y - x)), \mu(y)], \tag{4}$$

where S is a t-conorm and c a complementation. This corresponds to a degree of inclusion of ν, translated at x, in μ.

Let I bet a fuzzy implication. Fuzzy inclusion is related to implication by the following equation:

$$\mathcal{I}(\nu, \mu) = \inf_{x \in \mathcal{S}} I[\nu(x), \mu(x)]. \tag{5}$$

For the implication defined from a t-conorm (Eq. (1)), the erosion obtained from the derived inclusion is the one given by Eq. (4).

Instead of a t-conorm, a more general disjunction could be used, but this would lead to weaker properties. The result in Section 5 (Theorem 3) shows that S has to be a t-conorm if we want all usual properties of mathematical morphology to be satisfied.

The dual of erosion in the crisp case is $\delta_B(X) = (\varepsilon_{\check{B}}(X^c))^c$, where $\check{B}$ denotes the symmetrical of B with respect to the origin of $\mathcal{S}$. Accordingly, by duality with respect to the complementation c, fuzzy dilation is then defined by

$$\forall x \in \mathcal{S}, \quad \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} T[\nu(x-y), \mu(y)], \tag{6}$$

where T is the t-norm associated with the t-conorm S with respect to the complementation c. This definition of dilation corresponds to the translation of the following set equivalence:

$$x \in \delta_B(x) \;\Leftrightarrow\; \check{B}_x \cap X \neq \emptyset \;\Leftrightarrow\; \exists y \in \mathcal{S},\; y \in \check{B}_x \cap X.$$

The fuzzy dilation at x is expressed as the degree of intersection of ν translated at x and μ, which is dual of the degree of inclusion used for the erosion. If the t-norm is replaced by a general conjunction C, then weaker properties are derived, and again the result in Section 5 (Theorem 3) shows that T has to be a t-norm if we want all classical properties to hold.

These forms of fuzzy dilation and fuzzy erosion are very general, and several definitions found in the literature appear as particular cases, such as [9,10,1,27,31] (see e.g. [3,5,33] for a comparison). It is interesting to note that similar ideas have been developed independently at about the same time by different teams [2,9,31].

Finally, fuzzy opening (respectively, fuzzy closing) is simply defined as the combination $\delta\varepsilon$ (respectively, $\varepsilon\delta$) of a fuzzy erosion followed by a fuzzy dilation (respectively, a fuzzy dilation followed by a fuzzy erosion), by using dual t-norms and t-conorms.

The detailed properties of these definitions can be found in [5]. Most properties of classical morphology are satisfied whatever the choice of T and S. But in order to get true closing and opening, i.e. which are extensive (respectively, anti-extensive) and idempotent, a necessary and sufficient condition on T and S is

$$T(\beta, S(c(\beta), \alpha)) \leqslant \alpha, \tag{7}$$

which is satisfied for instance for Lukasiewicz t-norm and t-conorm (i.e. $T(\alpha, \beta) = \max(0, \alpha + \beta - 1)$ and $S(\alpha, \beta) = \min(1, \alpha + \beta)$) with the standard complementation ($c(\alpha) = 1 - \alpha$). Indeed, in this case $T(\beta, S(c(\beta), \alpha)) = \min(\alpha, \beta)$, which is always less than or equal to α. The property expressed by Eq. (7) can be found in [5,8].

3.2. Fuzzy morphology using adjunction and residual implications

A second type of approach is based on the notions of adjunction and fuzzy implication. Here the algebraic framework is the main guideline, which contrasts with the previous approach where duality was imposed in first place.

The derivation of fuzzy morphological operators from residual implication has first been proposed by De Baets [6,7], and then developed e.g. in [22,8]. One of its main advantages is that it leads to idempotent fuzzy closing and opening. This approach was formalized from the algebraic point of view of adjunction in [11]. It has then been used by other authors, e.g. [21]. This leads to general algebraic fuzzy erosions and dilations (i.e. operations that commute with the infimum and the supremum of the lattice, respectively). Let us detail this approach.

Let C be a conjunction, and I an implication. When using the implication in Eq. (2), the following expression for the degree of inclusion is derived: $\mathcal{I}(\nu, \mu) = \inf_{x \in \mathcal{S}} \sup\{\gamma \in [0,1], T(\nu(x), \gamma) \leqslant \mu(x)\}$. Fuzzy dilation and erosion are then defined as

$$\forall x \in \mathcal{S}, \quad \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} C(\nu(x-y), \mu(y)), \tag{8}$$

$$\forall x \in \mathcal{S}, \quad \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} I(\nu(y-x), \mu(y)). \tag{9}$$

Note that (I, C) is an adjunction if and only if $(\varepsilon_\nu, \delta_\nu)$ is an adjunction on the lattice $(\mathcal{F}, \leqslant)$ for any ν (i.e. $\delta_\nu(\mu) \leqslant \mu' \Leftrightarrow \mu \leqslant \varepsilon_\nu(\mu')$). Let us assume that (I, C) is an adjunction. Then

$$\begin{aligned}
\delta_\nu(\mu) \leqslant \mu' &\Leftrightarrow \forall\, x \in \mathcal{S}, \quad \sup_{y \in \mathcal{S}} C(\nu(x-y), \mu(y)) \leqslant \mu'(x) \\
&\Leftrightarrow \forall\, (x, y) \in \mathcal{S}^2, \quad C(\nu(x-y), \mu(y)) \leqslant \mu'(x) \\
&\Leftrightarrow \forall\, (x, y) \in \mathcal{S}^2, \quad \mu(y) \leqslant I(\nu(x-y), \mu'(x))
\end{aligned}$$

$$\Leftrightarrow \ \forall y \in \mathcal{S}, \quad \mu(y) \leqslant \inf_{x \in \mathcal{S}} I(v(x-y), \mu'(x))$$
$$\Leftrightarrow \ \forall y \in \mathcal{S}, \quad \mu(y) \leqslant \varepsilon_v(\mu')(y),$$

which proves that $(\varepsilon_v, \delta_v)$ is an adjunction This derivation shows by the way why the symmetrical of v has to be taken for one of the two operations. Conversely, if $(\varepsilon_v, \delta_v)$ is an adjunction, it is easy to show that (I, C) is an adjunction (by considering constant membership functions).

Opening and closing derived from these operations by combination have all required properties, whatever the choice of C and I. Some properties of dilation, such as iterativity, require C to be associative and commutative, i.e. a t-norm. This will be further investigated in Section 5.

Note that here the classical definition of adjunction is used. A fuzzification of this notion has been proposed in [23], based on the equality between the degree of inclusion of $\delta(\mu)$ in μ' and the degree of inclusion of μ in $\varepsilon(\mu')$. The degree of inclusion is defined as in Eq. (5), thus depending potentially on one point only, while keeping the general definition of adjunction imposes a constraint between dilation and erosion at every point of $\mathcal{S}$.

4. Links between both approaches

4.1. Dual vs. adjoint operators

A first trivial link between both approaches concerns the dilation: if C is a t-norm, then the dilation in the second approach is exactly the same as the one obtained in the first approach.

To understand further the relation between both approaches for erosion, let us denote by $\hat{I}$ the disjunction associated with an implication I and a negation (or complementation) c:

$$\hat{I}(\alpha, \beta) = I(c(\alpha), \beta).$$

Then $\hat{I}$ is increasing in both arguments, and if I is further assumed to satisfy $I(\alpha, \beta) = I(c(\beta), c(\alpha))$ (contraposition) and $I(\alpha, I(\beta, \gamma)) = I(\beta, I(\alpha, \gamma))$ (exchange principle), then $\hat{I}$ is commutative and associative, hence a t-conorm (we then recover the link between an implication I and a t-conorm S as in Eq. (1)).

Eq. (9) can be rewritten as

$$\varepsilon_v(\mu)(x) = \inf_{y \in \mathcal{S}} \hat{I}(c(v(y-x)), \mu(y)),$$

which corresponds to the fuzzy erosion of the first approach. The adjunction property can also be written as

$$C(\alpha, \beta) \leqslant \gamma \ \Leftrightarrow \ \beta \leqslant \hat{I}(c(\alpha), \gamma).$$

However, pairs of dual t-norms and t-conorms are not identical to pairs of adjoint operators. Let us take a few examples for the standard negation $c(a) = 1-a$, which is the most usual complementation. For $C = \min$, its adjoint is $I(\alpha, \beta) = \beta$ if $\beta < \alpha$, and 1 otherwise (known as Gödel implication). But the derived $\hat{I}$ is the dual of the conjunction defined as $C(\alpha, \beta) = 0$ if $\beta \leqslant 1-\alpha$ and β otherwise. Conversely, the adjoint of this conjunction is $I(\alpha, \beta) = \max(1-\alpha, \beta)$ (Kleene–Dienes implication), the dual of which is the minimum conjunction. Lukasiewicz operators $C(\alpha, \beta) = \max(0, \alpha+\beta-1)$ and $\hat{I}(\alpha, \beta) = \min(1, \alpha+\beta)$ are both adjoint and dual, which explains the exact correspondence between both approaches for these operators. Table 1 summarizes the differences between dual and adjoint operators for several examples.

Let us consider the last example, generated by a strictly increasing continuous mapping φ on [0, 1], with $\varphi(0) = 0$ and $\varphi(1) = 1$. The derived conjunction $T(\alpha, \beta) = \varphi^{-1}(\max(0, \varphi(\alpha) + \varphi(\beta) - 1))$ is the general form of continuous nilpotent t-norms. For the complementation defined by $c(\alpha) = \varphi^{-1}(1 - \varphi(\alpha))$ (with the same φ), the dual t-conorm writes

$$\begin{aligned} S(\alpha, \beta) &= c(T(c(\alpha), c(\beta))) \\ &= \varphi^{-1}(\min(1, \varphi(\alpha) + \varphi(\beta))). \end{aligned}$$

Notes that since φ^{-1} is increasing, $\varphi^{-1}(1) = 1$ and $\varphi^{-1}(0) = 0$, we can also write

$$T(\alpha, \beta) = \max(0, \varphi^{-1}(\varphi(\alpha) + \varphi(\beta) - 1))$$

Table 1
A few dual and adjoint operators: dual and adjoint are generally not identical, except in the case of Lukasiewicz operators and their generalized form.

Conjunction	Dual disjunction	Adjoint implication I	$\hat{I}$
$\min(\alpha, \beta)$	$\max(\alpha, \beta)$	$\begin{cases} \beta & \text{if } \beta < \alpha \\ 1 & \text{otherwise} \end{cases}$ (Gödel)	$\begin{cases} \beta & \text{if } \beta < 1-\alpha \\ 1 & \text{otherwise} \end{cases}$
$\begin{cases} 0 & \text{if } \beta \leqslant 1-\alpha \\ \beta & \text{otherwise} \end{cases}$	$\begin{cases} \beta & \text{if } \beta < 1-\alpha \\ 1 & \text{otherwise} \end{cases}$	$\max(1-\alpha, \beta)$ (Kleene–Dienes)	$\max(\alpha, \beta)$
$\alpha\beta$	$\alpha+\beta-\alpha\beta$	$\begin{cases} \dfrac{\beta}{\alpha} & \text{if } \beta \leqslant \alpha \text{ and } \alpha \neq 0 \\ 1 & \text{otherwise} \end{cases}$	$\begin{cases} \dfrac{\beta}{1-\alpha} & \text{if } \beta \leqslant 1-\alpha \text{ and } \alpha \neq 1 \\ 1 & \text{otherwise} \end{cases}$
$\begin{cases} 1-\dfrac{1-\beta}{\alpha} & \text{if } 1-\beta \leqslant \alpha \text{ and } \alpha \neq 0 \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \dfrac{\beta}{1-\alpha} & \text{if } \beta \leqslant 1-\alpha \text{ and } \alpha \neq 1 \\ 1 & \text{otherwise} \end{cases}$	$1-\alpha+\alpha\beta$	$\alpha+\beta-\alpha\beta$
$\max(0, \alpha+\beta-1)$	$\min(1, \alpha+\beta)$	$\min(1, 1-\alpha+\beta)$ (Lukasiewicz)	$\min(1, \alpha+\beta)$
$\varphi^{-1}(\max(0, \varphi(\alpha)+\varphi(\beta)-1))$	$\varphi^{-1}(\min(1, \varphi(\alpha)+\varphi(\beta)))$	$\varphi^{-1}(\min(1, 1-\varphi(\alpha)+\varphi(\beta)))$	$\varphi^{-1}(\min(1, \varphi(\alpha)+\varphi(\beta)))$

All examples are given for the standard negation, except the last one, in which $c(\alpha) = \varphi^{-1}(1-\varphi(\alpha))$, where φ is a strictly increasing continuous mapping on $[0, 1]$, with $\varphi(0) = 0$ and $\varphi(1) = 1$. The mapping φ is also the generator of these generalized Lukasiewicz operators.

and

$$S(\alpha, \beta) = \min(1, \varphi^{-1}(\varphi(\alpha) + \varphi(\beta))).$$

Let us now compute the residual implication, using the increasingness of φ:

$$\begin{aligned} I(\alpha, \beta) &= \sup\{\gamma \in [0, 1], T(\alpha, \gamma) \leqslant \beta\} \\ &= \varphi^{-1}(\min(1, 1-\varphi(\alpha)+\varphi(\beta))) = \min(1, \varphi^{-1}(1-\varphi(\alpha)+\varphi(\beta))). \end{aligned}$$

It is clear that we have $I(\alpha, \beta) = S(c(\alpha), \beta)$ (since $1-\varphi(\alpha) = \varphi(c(\alpha))$), hence these generalized Lukasiewicz operators are both adjoint and dual.

A particular case in this family is Schweizer and Sklar's operator [28], generated by $\varphi(\alpha) = \alpha^p$. The negation writes $c(\alpha) = (1-\alpha^p)^{1/p}$, the t-norm $T(\alpha, \beta) = \max(0, (\alpha^p + \beta^p - 1)^{1/p})$, the dual t-cornom $S(\alpha, \beta) = \min(1, (\alpha^p + \beta^p)^{1/p})$ and the residual implication $I(\alpha, \beta) = \min(1, ((1-\alpha)^p + \beta^p)^{1/p})$. It should be noted that it would not be possible to achieve both duality and adjunction for these operators by using another complementation, such as the standard one for instance. Another example is Yager's operator [34], generated by $\varphi(\alpha) = 1-(1-\alpha)^p$.

4.2. Equivalence condition

The first main result of this paper is expressed in the following theorem.

Theorem 1. *The condition for dual t-norms and t-conorms leading to idempotent opening and closing, given by Eq. (7) (i.e. $T(\beta, S(c(\beta), \alpha)) \leqslant \alpha$) is equivalent to the adjunction property between C and I for $T = C$ and $S = \hat{I}$.*

Proof. Let us assume that the adjunction property is satisfied for $T = C$ and $S = \hat{I}$, i.e.

$$T(\alpha, \beta) \leqslant \gamma \quad \Leftrightarrow \quad \beta \leqslant S(c(\alpha), \gamma). \tag{10}$$

Applying this property to the tautology $S(c(\beta), \alpha) \leqslant S(c(\beta), \alpha)$ leads directly to

$$T(\beta, S(c(\beta), \alpha)) \leqslant \alpha, \tag{11}$$

i.e. the condition in Eq. (7).

Let us now assume that we have the property expressed by Eq. (7) for dual operators.

- If $\beta \leqslant S(c(\alpha), \gamma)$, then since T is increasing, we have $T(\alpha, \beta) \leqslant T(\alpha, S(c(\alpha), \gamma))$ which is less than γ by Eq. (7). This implies $T(\alpha, \beta) \leqslant \gamma$.
- Eq. (7), expressed for $(c(\beta), \alpha)$ (instead of (α, β)), writes $T(\alpha, S(c(\alpha), c(\beta))) \leqslant c(\beta)$. Since T and S are dual, it is equivalent to $c(S(c(\alpha), c(S(c(\alpha), c(\beta))))) \leqslant c(\beta)$, hence $S(c(\alpha), T(\alpha, \beta)) \geqslant \beta$, since c is decreasing.

Now, if $T(\alpha, \beta) \leqslant \gamma$, since S is increasing, we have $S(c(\alpha), T(\alpha, \beta)) \leqslant S(c(\alpha), \gamma)$. Since the first term is greater than β, this implies $\beta \leqslant S(c(\alpha), \gamma)$. □

This result can also be derived, via the morphological operators, from the fact that if ε and δ are increasing operators such that $\varepsilon\delta$ is extensive and $\delta\varepsilon$ is anti-extensive, then (ε, δ) is an adjunction.[1] Conversely, if (ε, δ) is an adjunction, then the compositions $\varepsilon\delta$ and $\delta\varepsilon$ are extensive (respectively, anti-extensive) and idempotent, hence morphological closing and opening. This provides an algebraic and morphological proof of the same result, since Eq. (7) is a necessary and sufficient condition to have $\varepsilon\delta$ and $\delta\varepsilon$ idempotent and extensive (respectively, anti-extensive), and the adjunction property on (ε, δ) is equivalent to the adjunction property on (I, C).

Recent results [8,20] also show that a t-norm T is left continuous if and only if Eq. (7) holds, for S being derived from the residual implication I of the t-norm, and if and only if (T, I) is an adjunction, which shows the equivalence in the case of a residual implication derived from a left-continuous t-norm.

This result completes the link between both approaches by showing that duality and adjunction are generally not compatible, and that in case dual operators lead to true opening and closing, the condition on these operators is equivalent to the adjunction property. This means that in case duality and adjunction are compatible, the two approaches lead exactly to the same definitions. The following result expresses exactly when duality and adjunction are compatible.

Theorem 2. *Two continuous t-norm T and t-conorm S are both dual and adjoint (and the derived implication I is equivalently expressed from the t-conorm or from the t-norm by residuation) if and only if there exists a strictly continuous increasing mapping φ on $[0, 1]$, with $\varphi(0) = 0$ and $\varphi(1) = 1$, such that:*

$$T(\alpha, \beta) = \varphi^{-1}(\max(0, \varphi(\alpha) + \varphi(\beta) - 1)), \tag{12}$$
$$S(\alpha, \beta) = \varphi^{-1}(\min(1, \varphi(\alpha) + \varphi(\beta))), \tag{13}$$
$$I(\alpha, \beta) = \varphi^{-1}(\min(1, 1 - \varphi(\alpha) + \varphi(\beta))), \tag{14}$$

i.e. the operators are generalized Lukasiewicz operators.

Proof. This theorem directly results from general considerations on fuzzy implications. It has been proved by Smets and Magrez [32] that implications verifying a number of axioms are exactly generalized Lukasiewicz implications. These axioms are contraposition, exchange principle, monotonicity, boundary condition ($I(\alpha, \beta) = 1$ if and only if $\alpha \leqslant \beta$), neutrality ($I(1, \alpha) = \alpha$) and continuity.

Let us assume that we have both dual and adjoint operators. Most properties of the implication are directly derived from the properties of t-norms and t-conorms: contraposition is deduced from commutativity of the t-conorm, exchange principle from associativity, monotonicity from monotonicity, neutrality from the fact that 0 is the null element of t-conorms, continuity from continuity of t-norms and t-conorms in this case. The boundary condition is more interesting, since it is less trivially deduced from the adjunction property. Let us assume $I(\alpha, \beta) = 1$. Then, from the adjunction property, we have $T(\alpha, 1) \leqslant \beta$ and $\alpha \leqslant \beta$ from the fact that 1 is the null element of T. Conversely, if $\alpha \leqslant \beta$, then $T(\alpha, 1) \leqslant \beta$ and using the adjunction property $1 \leqslant I(\alpha, \beta)$. Hence $I(\alpha, \beta) = 1$. Therefore all Smets and Magrez' axioms are satisfied, showing that the only possibility is to have generalized Lukasiewicz operators. □

[1] Indeed, $\delta(x) \leqslant y \Rightarrow x \leqslant \varepsilon\delta(x) \leqslant \varepsilon(y)$ from the increasingness of ε and extensivity of $\varepsilon\delta$ and similarly $x \leqslant \varepsilon(y) \Rightarrow \delta(x) \leqslant \delta\varepsilon(y) \leqslant y$.

These results are in accordance with recently published results on fuzzy connectives [20], in particular concerning rotation-invariant left continuous t-norms, and are here put in the perspective of mathematical morphology. Note that the work in [20] provides very powerful characterizations in terms of "contour lines", with interesting geometric interpretations, as also in [18].

The computations in Section 4.1 show that duality and adjunction properties hold for these operators. It can also be checked that Eq. (7) holds:

$$\begin{aligned} T(\beta, S(c(\beta), \alpha)) &= T(\beta, S(\varphi^{-1}(1-\varphi(\beta)), \alpha)) \\ &= \max(0, \varphi^{-1}(\varphi(\beta) + \min(1, 1-\varphi(\beta)+\varphi(\alpha)) - 1)) \\ &= \max(0, \min(\alpha, \beta)) = \min(\alpha, \beta) \end{aligned}$$

which is always less than or equal to α.

4.3. Choice of the operators

The choice of the operators (conjunctions, t-conorms, implications) has a twofold influence, on theoretical and practical sides, and the understanding of this influence can guide the choice of a specific form of fuzzy mathematical morphology. From a theoretical point of view, when using adjoint operators, opening is a "true" opening (i.e. increasing, anti-extensive and idempotent) and closing is a true closing. The algebraic framework of mathematical morphology [26,30] can then be entirely applied in the fuzzy case, and a whole theory of morphological fuzzy filtering can then be directly derived. However, depending on the operators, some other properties of erosion and dilation can be lost, in particular if the chosen conjunction has not all properties of a t-norm. These aspects will be further investigated in Section 5. When using operators that are dual but not adjoint, opening is not anti-extensive, nor idempotent (but it is increasing).

From a practical point of view, two aspects can be considered: the properties that are actually needed for a specific application, and the spatial extent of the applied transformations (typically for image processing applications). For instance, duality may be of prime importance to be in agreement with set theoretical interpretations. This is the case for instance when considering a spatial object as a set (actually a fuzzy set) and the background as its complement. Iterativity of dilation and erosion is also important to guarantee that a sequence of operations has the desired behavior. It is also directly used when decomposing a complex structuring element into simpler parts, which has a direct impact on the algorithms and their computational complexity. Extensivity of dilation and anti-extensivity of erosion for a structuring element containing the origin of space is also an often required property. On the contrary, we may accept to loose some properties. For instance, if opening and closing are simply used to regularize the membership values and remove noisy points, even nonadjoint operators can be used (see for instance the examples in [5]). Concerning the spatial extent of the transformations, it is expected that it reflects the size of the structuring element. The monotonicity property of the main morphological operations guarantees that using a larger structuring element will have a larger effect on the transformed fuzzy set, thus preserving the order in some sense. But the extent of a transformation also depends on the choice of the operator: using smaller t-norms for dilation for instance will lead to smaller results as well. For instance, using the Lukasiewicz t-norm will lead to substantially smaller results than using the minimum (see the examples in [5,4]). In some cases, the results obtained with Lukasiewicz operators can be very close to the original fuzzy set, with an induced effect that appears smaller than the size of the structuring element.

Further examples can be found in [5,11,21,22], among others.

5. General forms of fuzzy morphological dilation and erosion

The second main result of this paper establishes the general form of fuzzy dilation and erosion, in order to satisfy a set of properties. Let $\delta_\nu(\mu)$ be a morphological dilation. Let us consider the following general form of δ:

$$\delta_\nu(\mu)(x) = g(f(\nu(x-y), \mu(y)), y \in \mathcal{S}), \tag{15}$$

where f is a mapping from $[0, 1] \times [0, 1]$ in $[0, 1]$ and g is a mapping from $[0, 1]^{\mathcal{S}}$ into $[0, 1]$ (the result is then a fuzzy set). Note that we do not rely here on any translation principle from the crisp case to the fuzzy case.

Theorem 3. *The compatibility of fuzzy dilation with classical dilation in case ν is crisp, its increasingness, and the commutativity with the supremum lead to the only possible form of δ*:

$$\delta_\nu(\mu)(x) = \sup_{y\in\mathcal{S}} C(\nu(x-y), \mu(y)),$$

where C is a conjunction. If the commutativity ($\delta_\nu(\mu) = \delta_\mu(\nu)$) and iterativity ($\delta_\nu\delta_{\nu'}(\mu) = \delta_{\delta_\nu(\nu')}(\mu)$) properties are also required, then C has to be a t-norm.

From this dilation, a unique erosion such that $(\varepsilon_\nu, \delta_\nu)$ is an adjunction is derived:

$$\varepsilon_\nu(\mu)(x) = \inf_{y\in\mathcal{S}} I(\nu(y-x), \mu(y)),$$

where I is the adjoint of C.

If duality is required, the disjunction $\hat{I}$ has to be the dual of C.

Proof. Let g_1 be the version of g applying on one variable only (i.e. if $\mathcal{S}$ contains only one point). It is a mapping from [0, 1] into [0, 1]. Most results are derived by considering constant membership functions. Increasingness of δ implies that the composition $g_1 f$ should be increasing in μ and ν. If ν is crisp, the compatibility with classical dilation implies that $\forall a \in [0,1],\ g_1(f(1,a)) = a$. Therefore $g_1 f$ is a conjunction.

Further properties such as commutativity and iterativity imply $g_1 f$ be commutative and associative, respectively, i.e. it should be a t-norm.

It is easy to prove that g_1 has to be a bijection (one-to-one mapping). It follows that $f(1,a) = g_1^{-1}(a)$. Let $\mu'(y) = g_1^{-1}(\mu(y))$. The compatibility with classical morphology implies $\sup_{y\in\mathcal{S}} \mu(y) = g(g_1^{-1}(\mu(y)), y \in \mathcal{S})$, i.e. $\sup_{y\in\mathcal{S}} g_1(\mu'(y)) = g(\mu'(y), y \in \mathcal{S})$. Therefore $\delta_\nu(\mu)(x) = \sup_{y\in\mathcal{S}} g_1(f(\nu(x-y), \mu(y)))$. From the properties of t-norms, this form commutes with the supremum.

From a dilation δ_ν, a general result on adjunctions [17] guarantees that there exists a unique erosion ε_ν such that $(\varepsilon_\nu, \delta_\nu)$ is an adjunction, and it is given by

$$\varepsilon_\nu(\mu) = \bigvee\{\mu' \in \mathcal{F}, \delta_\nu(\mu') \leqslant \mu\}.$$

We have the following equivalences, by denoting $g_1 f = T$ and I the adjoint of T:

$$\begin{aligned}
\delta_\nu(\mu') \leqslant \mu \;&\Leftrightarrow\; \forall x \in \mathcal{S},\quad \delta_\nu(\mu')(x) \leqslant \mu(x)\\
&\Leftrightarrow\; \forall (x,y) \in \mathcal{S}^2,\quad T(\nu(x-y), \mu'(y)) \leqslant \mu(x)\\
&\Leftrightarrow\; \forall (x,y) \in \mathcal{S}^2,\quad \mu'(y) \leqslant I(\nu(x-y), \mu(x))\\
&\Leftrightarrow\; \forall y \in \mathcal{S},\quad \mu'(y) \leqslant \inf_{x\in\mathcal{S}} I(\nu(x-y), \mu(x)).
\end{aligned}$$

Since ε_ν is the supremum of μ' verifying this equation, we have: $\varepsilon_\nu(\mu)(y) = \inf_{x\in\mathcal{S}} I(\nu(x-y), \mu(x))$.

Now, if duality is required between ε_ν and δ_ν with respect to complementation, it is straightforward to show that T and $\hat{I}$ have to be dual operators.

Having both duality and adjunction is possible under the conditions expressed in Theorem 1. Furthermore Theorem 2 provides the general form of the corresponding operators. □

In [31], a similar approach was developed for deriving a general form of fuzzy inclusion (from which fuzzy erosion is derived). Since weaker properties are required, this approach leads to the use of weak t-norms and t-conorms (they are not associative and do not admit 1 (respectively 0) as unit element, in general). Properties of morphological operators are then weaker (no iterativity can be expected, no compatibility with classical morphology), and this is therefore somewhat less interesting from a morphological point of view. Our approach overcomes these drawbacks.

6. Conclusion

This paper exhibits the exact conditions to have a convergence between the two main approaches for fuzzy morphology. Although the underlying principles are not compatible in general, it is interesting to note that in case they are consistent, then both approaches are equivalent. The appropriate class of operators satisfying all required conditions is exactly constituted by the generalized Lukasiewicz operators. Furthermore, they provide the most general forms in order to satisfy a set of reasonable properties as in classical morphology. These two new results clarify the status of different forms of mathematical morphology.

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