Chapter 14

MATHEMATICAL MORPHOLOGY

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1. Introduction

Mathematical morphology (MM) is a branch of image processing, which arose in 1964. It is associated with the names of Georges Matheron and Jean Serra, who developed its main concepts and tools, expounded it in several books (Matheron, 1975; Serra, 1982; Serra, 1988), and created a team at the *Centre de Morphologie Mathématique* on the Fontainebleau site of the Paris School of Mines.

MM truly deserves the adjective "mathematical", as it is heavily mathematized. In this respect, it contrasts with the various heuristic or experimental approaches to image processing that one sees in the literature. It stands also as an alternative to another strongly mathematized branch of image processing, the one that bases itself on signal processing and information theory, following the works of prestigious pioneers named Wiener, Shannon, Gabor, etc. Indeed, these classical approaches proved their value in telecommunications. However

MM claims that analysing the information of an image is not like transmitting a signal on a channel, that an image should not be considered as a combination of sinusoidal frequencies, nor as the result of a Markov process on individual points. It considers that the purpose of image analysis is to find spatial objects, therefore images contain geometrical shapes with luminance (or colour) profiles, which can be investigated by their interactions with other shapes and luminance profiles. This makes the morphological approach especially relevant in situations where image grey-levels (or colours) correspond directly to significant material data, as in medical imaging, microscopy, industrial inspection and remote sensing.

In its development, MM has borrowed concepts and tools from various branches of mathematics: algebra (lattice theory), topology, discrete geometry, integral geometry, geometrical probability, partial differential equations, etc.; in fact any mathematical theory that deals with shapes, their combinations or their evolution, can be brought to contribute to morphological theory.

MM started by analysing binary images (sets of points) with the use of set-theoretical operations. In order to apply it to other types of images, for example grey-level ones (numerical functions), it was necessary to generalize set-theoretical notions, such as the relation of inclusion and the operations of union and intersection. This was done by using the lattice-theoretical notions of a *partial order* relation between images, for which the operations of *supremum* (least upper bound) and *infimum* (greatest lower bound) are defined. Therefore the central structure in MM is that of a *complete lattice*, and the basic morphological operators (dilation, erosion, opening and closing) can be characterized in this framework.

When analysing sets, one considers their topology: is the set in one or several pieces, how many holes has it, etc. Some topological notions, in particular connectedness, have been generalized in the framework of complete lattices. Nowadays, most morphological techniques combine lattice-theoretical and topological methods.

The computer processing of pictures quickly led to digital models of geometry. The pioneering work in this field is that of Azriel Rosenfeld, who died in 2004 after having contributed to digital geometry and image processing for 40 years. Thanks to its algebraic formalism, mathematical morphology is perfectly adapted to the digital framework. Moreover, the topology of digital figures can be studied in the framework of *combinatorial topology*, a field that was developed in the first half of the 20th century by mathematicians like Paul Alexandroff (Alexandroff, 1937; Alexandroff, 1956; Alexandroff and Hopf, 1935). In particular the latter proposed in 1935 to subdivide the Euclidean plane into rectangular cells, in such a way that cell interiors, sides, and corners are considered as points in an abstract space, whose combinatorial relations provide the topology. This idea prefigured the notion of pixels, and the corresponding Alexandroff topology was formally developed by Efim Khalimsky and popularized by Vladimir Kovalevsky; it has been shown that many "paradoxes" of digital geometry (like non-parallel lines which do not intersect) find a natural solution in that topology.

MM has also borrowed tools from integral geometry in order to measure some parameters on images. However these measurements are usually preceded by some image processing operations, in order to restrict the measure to some appropriate features: for example, to estimate the average length of particles whose width is at least w, we apply first an operator eliminating all particles narrower than w, then we make a length measurement on the remaining ones.

MM has also a probabilistic aspect, where images and shapes can be considered as random events. Suppose for example that one asks n experts to extract a certain set S from an image, say n anatomists have to extract the left half of the liver from an X-ray scanner hepatic image; they will disagree, and extract n different sets $S_1, \ldots, S_n$; now, how does one derive the "average" of these n sets, or their "standard deviation"? Furthermore, if one designs a computer algorithm for extracting that set, which produces the set S_{auto}, how does on evaluate the statistical significance of S_{auto} w.r.t. to the distribution $S_1, \ldots, S_n$? Such problems are studied in geometric probability, through the theory of random sets and functions (Matheron, 1975; Serra, 1982; Serra, 1988). This should not be confused with Markov field models for image processing: there the random variable is the grey-level of an individual pixel, and it evolves in space by a Markov process.

Image analysis has considered the varying scales at which things are seen. This has been formalized by multi-scale models governed by partial differential equations (PDEs). This has happened also for morphological operators, for which new PDEs have been given, leading to a new understanding of their functioning.

The theory of morphological operators relies on the formalism of lattice theory, and the latter underlies also several theoretical aspects of computer science: fuzzy sets, formal concept analysis and abstract interpretation of programming. In fact, the lattice-theoretical tools developed in each speciality can be used for the other ones. For example, a research on fuzzy morphology has been undertaken since several years. Also, the tools of MM, developed for the purpose of filtering and segmenting images, have found applications for modelling spatial concepts, like "close to" or "between".

The link between logic and lattice theory is obvious. Boole's logic is the first example of a Boolean algebra, while non-classical logics have been modeled as non-Boolean lattices. As MM analyses spatial shapes by means of lattice-theoretical operations, it is adapted to the logical analysis of spatial relations, while its abstract mathematical tools can be used in order to illuminate some aspects of logic, for example modal logic, and to build new operations in such a framework.

The purpose of this chapter is to present the basic theory of MM (Sec. 2), then to show how its tools can be applied to various specialities dealing with the analysis of spatial shapes and spatial relations, such as formal concept analysis, rough sets and fuzzy sets (Sec. 3), and finally to show its relevance in logic (Sec. 4).

Let us now describe the basic operations of mathematical morphology, first in the case of sets (or binary images), and next in the case of numerical functions (or grey-level images). We must warn the reader that in several works (including important ones, for instance Serra, 1982; Soille, 2003), the definitions given for the basic operations (Minkowski addition and subtraction, dilation, erosion, opening and closing) differ from ours in that in some cases the structuring element must be replaced by its symmetrical; also the notation can be different (in particular Serra, 1982). The definitions given here for morphological operations are standard (Heijmans, 1994), in the sense that they are consistent with the original definitions given by Minkowski, 1903 for the Minkowski addition and Hadwiger, 1950 for the Minkowski subtraction, and that they follow the algebraic theory (see Sec. 2), which allows to give a unified treatment (Heijmans and Ronse, 1990; Ronse and Heijmans, 1991) of such operators in the case of sets, numerical functions, and many other structures.

1.1 Morphology on sets

Consider the space $E = \mathbb{R}^n$ or $\mathbb{Z}^n$, with origin $o = (0, \ldots, 0)$. Given $X \subseteq E$, the *complement* of $X \subseteq E$ is $X^c = E \setminus X$, and the *transpose* or *symmetrical* of X is $\check{X} = \{-x \mid x \in X\}$. For every $p \in E$, the *translation* by p is the map $E \to E : x \mapsto x + p$; it transforms any subset X of E into its *translate by* p, $X_p = \{x + p \mid x \in X\}$.

Most morphological operations on sets can be obtained by combining set-theoretical operations with two basic operators, *dilation* and *erosion*. The latter arise from two set-theoretical operations, the *Minkowski addition* $\oplus$ (Minkowski, 1903) and *subtraction* $\ominus$ (Hadwiger, 1950), defined as follows for any $X, B \in \mathcal{P}(E)$:

$$
\begin{aligned}
X \oplus B &= \bigcup_{b \in B} X_b \;, \\
&= \bigcup_{x \in X} B_x \;, \\
&= \{x + b \mid x \in X, b \in B\} \;; \\
X \ominus B &= \bigcap_{b \in B} X_{-b} \;, \\
&= \{p \in E \mid B_p \subseteq X\} \;.
\end{aligned}
\tag{14.1}
$$

Formally speaking, X and B play similar roles as binary operands. However, in real situations, X will stand for the *image* (which is big, and given by the problem), and B for the *structuring element* (a small shape chosen by the user), so that $X \oplus B$ and $X \ominus B$ will be transformed images. We define the *dilation by* B, $\delta_B : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto X \oplus B$, and the *erosion by* B, $\varepsilon_B : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto X \ominus B$. It should be noted that dilation and erosion are *dual by complementation*, in other words dilating a set is equivalent to eroding its complement with the symmetrical structuring element:

$$(X \oplus B)^c = X^c \ominus \check{B} \ , \qquad (X \ominus B)^c = X^c \oplus \check{B} \ . \tag{14.2}$$

Therefore the properties of erosion are derived from those of dilation by duality: dilation inflates the object, deflates the background and deforms convex corners of the object; thus erosion deflates the object, inflates the background and deforms concave corners of the object. By Equation (14.2), we can also obtain alternate formulations for Minkowski addition and subtraction:

$$\begin{array}{rcl} X \oplus B & = & \{p \in E \mid (\check{B})_p \cap X \neq \emptyset\} \ ; \\ X \ominus B & = & \{p \in E \mid \forall z \notin X, p \notin (\check{B})_z\} \ . \end{array} \tag{14.3}$$

We illustrate in Fig. 14.1 the dilation and erosion of a cross by a triangular structuring element.

Dilation and erosion are the basic elements from which most morphological operators are built. The first example is the *hit-or-miss transform*, which uses a pair of structuring elements. Let A and B be two disjoint subsets of E; A will be the *foreground structuring element* and B the *background structuring element*; we then define:

$$\begin{array}{rcl} X \circledast (A, B) & = & \{p \in E \mid A_p \subseteq X \text{ and } B_p \subseteq X^c\} \ , \\ & = & (X \ominus A) \cap (X^c \ominus B) = (X \ominus A) \setminus (X \oplus \check{B}) \ . \end{array}$$

This will give the locus of all points where A fits the foreground and B fits the background. This operation corresponds to what is usually called *template matching*.

The main operators derived from dilation and erosion are *opening* and *closing*. We define the binary operations $\circ$ and $\bullet$ by setting for any $X, B \in \mathcal{P}(E)$:

$$\begin{array}{rcl} X \circ B & = & (X \ominus B) \oplus B \ , \\ & = & \bigcup\{B_p \mid p \in E \text{ and } B_p \subseteq X\} \ ; \\ X \bullet B & = & (X \oplus B) \ominus B \ . \end{array} \tag{14.4}$$

The operator $\gamma_B : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto X \circ B$ is called the *opening by* B; it is the composition of the erosion ε_B, followed by the dilation δ_B. On the other hand, the operator $\varphi_B : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto X \bullet B$ is called the *closing*

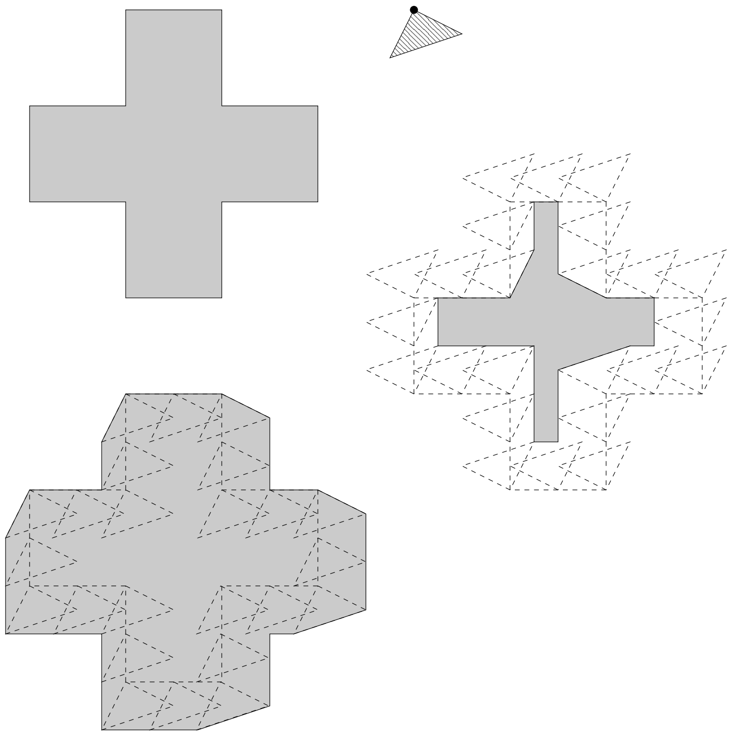


Figure 14.1. Top: The figure X is the cross, and the structuring element B is the triangle; the position of the origin is indicated by a thick dot. Bottom: The dilation $X \oplus B$ of X by B. Right: The erosion $X \ominus B$ of X by B is obtained as the complement of the dilation $X^c \oplus \check{B}$ of the complement X^c by the symmetrical structuring element $\check{B}$.

by B; it is the composition of the dilation δ_B, followed by the erosion ε_B. The two are dual by complementation:

$$(X \circ B)^c = X^c \bullet \check{B} \ , \qquad (X \bullet B)^c = X^c \circ \check{B} \ . \tag{14.5}$$

Hence the properties of closing are derived from those of opening by duality: opening removes narrow parts of the object and deforms convex corners of the object; thus closing fills narrow parts of the background and deforms concave corners of the object. We illustrate in Fig. 14.2 the opening and closing of a cross by a triangular structuring element.

Given a family $\mathcal{B}$ of structuring elements, the opening by $\mathcal{B}$, written $\gamma_{\mathcal{B}}$, is the union of openings by elements of $\mathcal{B}$, while the closing by $\mathcal{B}$, written $\varphi_{\mathcal{B}}$, is

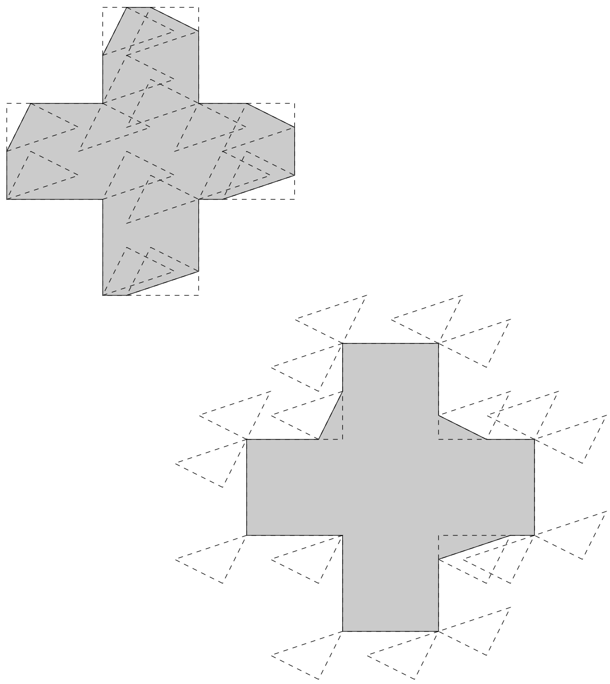


Figure 14.2. We use the same figure X and structuring element B as in Fig. 14.1. Top left: The opening $X \circ B$ of X by B is the union of all translates of B which are included in X. Bottom right: The closing $X \bullet B$ of X by B is obtained as the complement of the opening $X^c \circ \check{B}$ of the complement X^c by the symmetrical structuring element $\check{B}$.

the intersection of closings by elements of $\mathcal{B}$:

$$\begin{aligned} \gamma_{\mathcal{B}}(X) &= \bigcup_{B\in\mathcal{B}} (X \circ B) \ , \\ \varphi_{\mathcal{B}}(X) &= \bigcap_{B\in\mathcal{B}} (X \bullet B) \ . \end{aligned} \tag{14.6}$$

For example, if H and V are respectively a horizontal and a vertical line segment of length a, $\gamma_{\{H,V\}}$ will extract from a line drawing all horizontal and vertical lines of length at least a (as well as all blobs whose height or width is at least a).

The most interesting properties of the opening and closing (by one or several structuring elements) is that they are *idempotent*: $\gamma_{\mathcal{B}}(\gamma_{\mathcal{B}}(X)) = \gamma_{\mathcal{B}}(X)$ and $\varphi_{\mathcal{B}}(\varphi_{\mathcal{B}}(X)) = \varphi_{\mathcal{B}}(X)$. This means that if we consider them as filters, they do their job completely, and there is no need to repeat them. This contrasts with the behaviour of other image processing operators, like the median filter, where repeated applications can further modify the image, without a guarantee that it will reach a stable result after a finite number of iterations (indeed, the median filter can produce oscillations). The opening can be used to filter out *positive noise*, that is, to remove noisy parts of the object, typically small components; on the other hand, the closing can be used to remove *negative noise*, that is, to add to the object noisy parts of the background, typically small holes. By repeated composition of an opening and a closing, one can obtain four new filters:

- opening followed by closing;
- closing followed by opening;
- opening followed by closing, then by opening;
- closing followed by opening, then by closing.

All four are idempotent, and no other operator can be obtained by further composition (Serra, 1988). They can be used as filters to remove both positive and negative noise; for example, they constitute an alternative to median filtering for removing speckle noise.

These operators have a drawback: they deform the frontier between the object and background. Typically, if one uses a disk-shaped structuring element, they will round the corners of objects. However, one may want to filter out small components or holes of the object, without modifying the shape of the other components and holes. In other words, we look for filters which do not act at the level of pixels, but of connected components of the foreground (called *grains*) and of the background (called *pores*).

The basic operation for this purpose is shown in Fig. 14.3: from a figure F, we extract the union of all connected components of F (grains) that intersect a marker R.

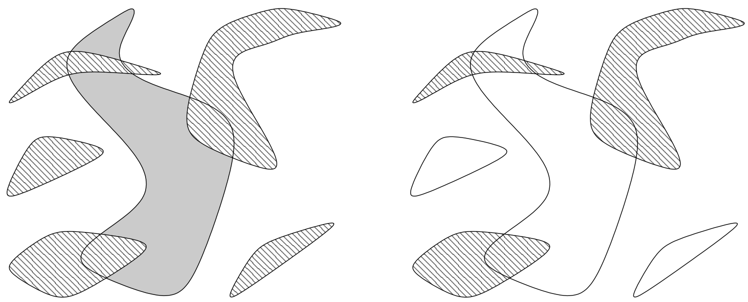


Figure 14.3. Left: We have a figure F (shown hatched) and a marker R (grey). Right: All connected components of F that intersect R are shown hatched.

We can formalize this operation as follows. We assume that E is a digital space ($E = \mathbb{Z}^n$ or a bounded grid in $\mathbb{Z}^n$), and that the connectivity arises from an adjacency graph on E, for example, the 4- or 8-adjacency on $\mathbb{Z}^2$, the 6- or 26-adjacency on $\mathbb{Z}^3$ (Rosenfeld and Kak, 1976). Let V be the structuring element comprising the origin o and the pixels adjacent to it, so that for any pixel p, the set comprising p and its neighbours is V_p; note that V is symmetrical ($V = \check{V}$). Given a set F (called the *mask*) and a subset R of F (called the *marker*), we define the *geodesical reconstruction by dilation* (from marker R in the mask F) as the limit

$$rec_{\oplus}(F, R) = \bigcup_{n \in \mathbb{N}} R_n$$

of the increasing sequence of sets R_n, $n \in \mathbb{N}$, defined recursively as follows:

$$R_0 = R \cap F \quad \text{and} \quad \forall n \in \mathbb{N}, \quad R_{n+1} = (R_n \oplus V) \cap F \ .$$

This will indeed give the union of all grains of F *marked by* (i.e., intersecting) the marker R.

The dual operation is the *geodesical reconstruction by erosion*; here the marker R is a superset of the mask F ($F \subseteq R$), and it is defined as

$$rec_{\ominus}(F, R) = \left[rec_{\oplus}(F^c, R^c) \right]^c \ .$$

This is in fact the limit $\bigcap_{n \in \mathbb{N}} R_n$ of the decreasing sequence of sets R_n, $n \in \mathbb{N}$, defined recursively by

$$R_0 = R \cup F \quad \text{and} \quad \forall n \in \mathbb{N}, \quad R_{n+1} = (R_n \ominus V) \cup F \ .$$

The behaviour of $rec_{\ominus}$ is to reconstruct all pores of F which are not completely covered by the marker R; in other words, all connected components of the background F^c which are included in R, are added to F. We illustrate this operation in Fig. 14.4.

Given an opening γ, we define the *opening by reconstruction* γ_{rec} as the geodesical reconstruction by dilation using the opening as marker:

$$\gamma_{rec}(X) = rec_{\oplus}(X, \gamma(X)) \ .$$

Similarly for a closing φ, we define the *closing by reconstruction* φ_{rec} as the geodesical reconstruction by erosion using the closing as marker:

$$\varphi_{rec}(X) = rec_{\ominus}(X, \varphi(X)) \ .$$

Note that for a connected structuring element B containing the origin, we have

$$\begin{aligned} rec_{\oplus}(X, X \circ B) &= rec_{\oplus}(X, X \ominus B) \\ \text{and} \quad rec_{\ominus}(X, X \bullet B) &= rec_{\ominus}(X, X \oplus B) \ . \end{aligned}$$

The opening and closing by reconstruction are again idempotent operators; they respectively remove small grains and fill small pores, but they do not deform the

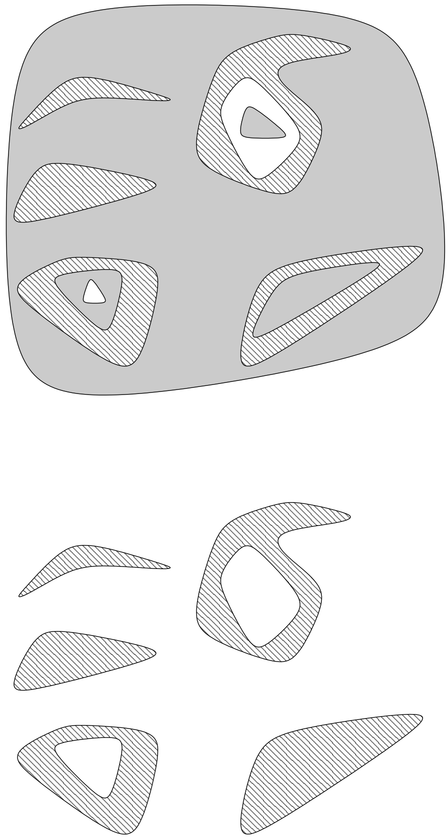


Figure 14.4. Top: We have a mask figure F (hatched) contained in a marker R ($R \setminus F$ is shown in grey). Bottom: $rec_{\ominus}(F, R)$ (hatched) is made of F and all connected components of F^c which are completely covered by the marker R.

remaining boundaries between foreground and background. They can then be composed (as explained above: opening followed by closing, closing followed by opening, etc.) in order to provide idempotent filters that remove grains and pores on the basis of their width, without distorting the contours of objects.

Other idempotent filters can be built, that act directly on grains and pores, for example, the *area opening* (which removes all grains whose area is below a threshold) and the *area closing* (filling all pores whose area is below a threshold).

1.2 Morphology on functions

In computer imaging, grey-levels are coded by numerical values, the low ones corresponding to dark pixels, and the high ones corresponding to bright ones. Hence in mathematical morphology (Heijmans, 1994), grey-level images are usually considered as numerical functions $E \to T$, where E is the space of points and T is the set of grey-levels; it is always a subset of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The grey-levels are numerically ordered, and morphological operations usually compute at each point in E a combination of numerical suprema and infima of grey-level values. Thus one supposes that T is closed under the operations of non-empty numerical supremum and infimum; in the terminology that we will introduce in Sec. 2, T is a *complete lattice*. Usually one takes for T one of the sets $\overline{\mathbb{R}}$, $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, $[a, b] = \{x \in \overline{\mathbb{R}} \mid a \leq x \leq b\}$ (with $a, b \in \overline{\mathbb{R}}$ and $a < b$), or $[a \ldots b] = [a, b] \cap \overline{\mathbb{Z}}$ (with $a, b \in \overline{\mathbb{Z}}$ and $a < b$). We write t_0 and t_1 respectively for the least and greatest element of T (thus $t_0 = -\infty$ and $t_1 = +\infty$ for $T = \overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$, while $t_0 = a$ and $t_1 = b$ for $T = [a, b]$ or $[a \ldots b]$).

The set T^E of functions $E \to T$ inherits the numerical order on T by the pointwise ordering of functions:

$$F \leq G \qquad \Longleftrightarrow \qquad \forall p \in E, \quad F(p) \leq G(p) \ . \tag{14.7}$$

This is the analogue for functions of the inclusion relation for sets. Now the analogues for functions of the union and intersection operations for sets, are the *supremum* (least upper bound) and *infimum* (greatest lower bound), obtained by pointwise supremum and infimum operations:

$$\bigvee_{i \in I} F_i : E \to T : p \mapsto \sup_{i \in I} F_i(p) \ , \ \bigwedge_{i \in I} F_i : E \to T : p \mapsto \inf_{i \in I} F_i(p) \ . \tag{14.8}$$

We write $F \vee G$ and $F \wedge G$ for the supremum and infimum of two functions (cf. the union and intersection of two sets); as the two binary operations $\vee$ and $\wedge$ are commutative and associative, we can write $F_1 \vee \cdots \vee F_n$ and $F_1 \wedge \cdots \wedge F_n$, which are in fact respectively equal to $\bigvee_{i \in \{1,\ldots,n\}} F_i$ and $\bigwedge_{i \in \{1,\ldots,n\}} F_i$. The least and greatest functions are the ones with constant values t_0 and t_1 respectively, they are the analogues of the empty set $\emptyset$ and the whole space E.

Given a function $F : E \to T$ and a point $p \in E$, the *translate of F by p* is the function F_p whose graph is obtained by translating the graph $\{(x, F(x)) \mid x \in E\}$ by p in the first coordinate, that is,

$$\{(y, F_p(y)) \mid p \in E\} = \{(x + p, F(x)) \mid x \in E\} \ ,$$

in other words

$$\forall y \in E, \qquad F_p(y) = F(y - p) \ .$$

We have thus the analogues for functions of the union, intersection and translation operations for sets. It is then possible to define the dilation, erosion, opening and closing of a function by a structuring element, by making analogues of Eqs. (14.1,14.4).

There is however a systematic method for extending operators on sets to operators on functions (Heijmans, 1991; Heijmans, 1994; Ronse, 2003). It relies on the notions of *thresholding* and *stacking*. Given a function $F : E \to T$, the *umbra* (or *hypograph*) of F is the set

$$U(F) = \{(p,t) \mid p \in E, t \in T, F(p) \geq t\} \ ,$$

and for any value $t \in T$, consider the *threshold set*

$$X_t(F) = \{p \in E \mid F(p) \geq t\} \ ;$$

thus $(p,t) \in U(F)$ iff $p \in X_t(F)$. We illustrate these notions in Fig. 14.5.

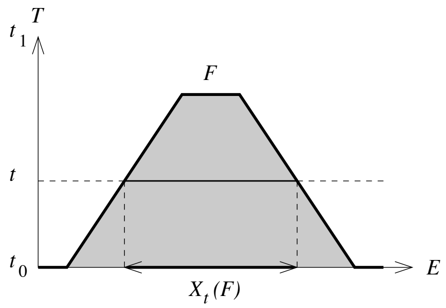


Figure 14.5. The graph of F, and below it the umbra $U(F)$ (in grey). For $t \in T$, the horizontal line at level t crosses the umbra in a section whose projection in E is the threshold set $X_t(F)$.

Given an operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$, the *flat operator corresponding to* ψ (or *flat extension of* ψ) is the operator $\psi^T : T^E \to T^E$ constructed as follows:

1 *Thresholding:* For every $t \in T$, we take the horizontal cross-section of the umbra $U(F)$ at level t, that is the set $X_t(F) \times \{t\}$.

2 *Horizontal operation:* We apply ψ horizontally to every such cross-section, that is, for every $t \in T$ we obtain the set $\psi\left(X_t(F)\right) \times \{t\}$.

3 *Stacking:* The upper envelope of these sets $\psi\left(X_t(F)\right) \times \{t\}$, $t \in T$, defines a function which gives $\psi^T(F)$.

We illustrate this construction in Fig. 14.6, in the case where $\psi = \delta_B$, the dilation by a structuring element B. In fact, the values taken by $\psi^T(F)$ are given by the following formula:

$$\text{(14.9)} \qquad \forall p \in E, \quad \psi^T(F)(p) = \bigvee \{t \in T \mid p \in \psi\left(X_t(F)\right)\} \ .$$

Rather than using Equation (14.9) to compute the values $\psi^T(F)(p)$, we can rely on the fact that the flat extension of operators transforms the operations on

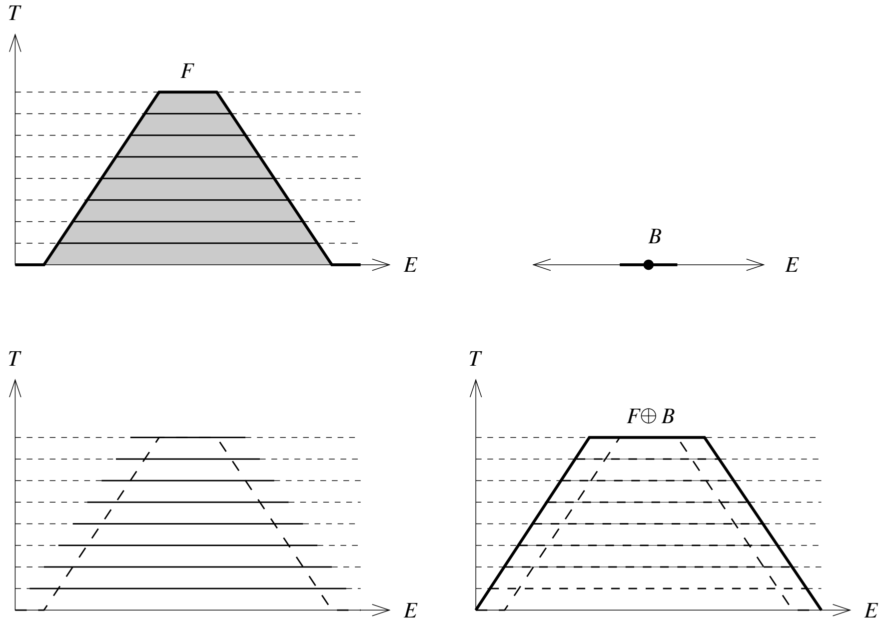


Figure 14.6. Top left: The graph of F, the umbra $U(F)$ (in grey), and horizontal cross-sections $X_t(F) \times \{t\}$ of the umbra. Top right: The structuring element B (the position of the origin is indicated by a dot). Bottom left: We apply δ_B, the dilation by B, horizontally to the cross-sections $X_t(F) \times \{t\}$, obtaining the sets $(X_t(F) \oplus B) \times \{t\}$. Bottom right: The upper envelope of the dilated cross-sections gives the dilated function $\delta_B^T(F)$, also written $F \oplus B$.

sets into the corresponding ones on functions, as it follows from the properties listed below (for the sake of brevity, in the formulas we omit the quantifications $\forall X \in \mathcal{P}(E)$ and $\forall F \in T^E$):

- *Identity*: If $\psi(X) = X$, then $\psi^T(F) = F$.
- *Translation*: If $\psi(X) = X_p$, then $\psi^T(F) = F_p$.
- *Union*: If $\psi(X) = \bigcup_{i\in I} \xi_i(X)$, then $\psi^T(F) = \bigvee_{i\in I} \xi_i^T(F)$.
- *Intersection*: If $\psi(X) = \bigcap_{i\in I} \xi_i(X)$, then $\psi^T(F) = \bigwedge_{i\in I} \xi_i^T(F)$.
- *Composition*: If $\psi(X) = \eta(\zeta(X))$, then $\psi^T(F) = \eta^T(\zeta^T(F))$.

These properties can for example be used to give formulas for the flat extensions of dilation and erosion. As $\delta_B(X) = \bigcup_{b\in B} X_b$ and $\varepsilon_B(X) = \bigcap_{b\in B} X_{-b}$ (see Equation (14.1)), we obtain for every $F \in T^E$:

(14.10) $$\delta_B^T(F) = \bigvee_{b\in B} F_b \quad \text{and} \quad \varepsilon_B^T(F) = \bigwedge_{b\in B} F_{-b} \ .$$

We get then for every $p \in E$:

$$(14.11)\qquad \begin{array}{rrcccl} & \delta_B^T(F)(p) & = & \sup_{b\in B} F(p-b) & = & \sup_{q\in(\check{B})_p} F(q) \\ \text{and} & \varepsilon_B^T(F)(p) & = & \inf_{b\in B} F(p+b) & = & \inf_{q\in B_p} F(q)\ . \end{array}$$

It is customary to write $F \oplus B$ and $F \ominus B$ for $\delta_B^T(F)$ and $\varepsilon_B^T(F)$. Following Equation (14.4), we define $F \circ B = (F \ominus B) \oplus B$ and $F \bullet B = (F \oplus B) \ominus B$; clearly $F \circ B = \gamma_B^T(F)$ and $F \bullet B = \varphi_B^T(F)$. Note that here the operations $\oplus$, $\ominus$, $\circ$ and $\bullet$ have a function as first operand, a set as second, and a function again as result.

All set operators built by combining dilations and erosions through unions, intersections and translations, extend thus naturally as *flat operators*. Then the properties of the set operators translate directly to their flat extensions; for example, openings and closings are idempotent, and composing them leads to idempotent filters. In practice, flat operators behave on bright and dark parts of a grey-level image in the same way as the corresponding set operators do on foreground and background. For example, dilation inflates bright areas and deflates dark ones, while erosion does the contrary; opening darkens narrow bright zones, while closing brightens narrow dark zones; dilation and opening deform corners which are convex on the bright side, while erosion and closing deform corners which are convex on the dark side. In particular, filters obtained by composing opening and closing can be used to remove small defects in an image, such as speckle noise.

There is still a duality between erosion and dilation, and between opening and closing. Let n be an *inversion* of T, that is a bijection $T \to T$ which reverses the order: $t < t' \iff n(t) > n(t')$; for example, if $T = [a \ldots b]$, we have $n(t) = a + b - t$; we extend it to an inversion N on functions, by setting $N(F) : p \mapsto n(F(p))$ (here n and N stand for *negative*, in the photographic sense). Then:

$$\begin{array}{rclrcl} N\big(F \oplus B\big) & = & N(F) \ominus \check{B}\ , & N\big(F \ominus B\big) & = & N(F) \oplus \check{B}\ , \\ N\big(F \circ B\big) & = & N(F) \bullet \check{B}\ , & N\big(F \bullet B\big) & = & N(F) \circ \check{B}\ . \end{array}$$

This expresses formally the fact that the behaviour of erosions and closings is derived of that of dilations and openings, by exchanging the roles of bright and dark points or zones in the grey-level image.

It is also possible to give flat extensions of geodesical reconstruction by dilation or erosion. For a *mask function* and a *marker function* R, such that $R \leq F$, we define the *geodesical reconstruction by dilation*

$$rec_{\oplus}(F, R) = \bigvee_{n\in\mathbb{N}} R_n\ ,$$

where the functions R_n, $n \in \mathbb{N}$, are defined recursively by

$$R_0 = R \wedge F \qquad \text{and} \qquad \forall n \in \mathbb{N},\quad R_{n+1} = (R_n \oplus V) \wedge F\ .$$

(Here V is the neighbourhood of the origin.) For $R \geq F$, we have the *geodesical reconstruction by erosion*

$$rec_{\ominus}(F, R) = \bigwedge_{n \in \mathbb{N}} R_n \ ,$$

where

$$R_0 = R \vee F \qquad \text{and} \qquad \forall n \in \mathbb{N}, \quad R_{n+1} = (R_n \ominus V) \vee F \ .$$

In fact, the two are dual:

$$rec_{\ominus}(F, R) = N\big[rec_{\oplus}\big(N(F), N(R)\big)\big] \ .$$

In the same way as the geodesical reconstructions on sets acted on grains and pores (connected components of the foreground and background), here these operators will act on *flat zones*, that is, maximal connected sets having a constant grey-level value. In particular, we can design openings and closings by reconstruction, as in the case of sets, and these filters will remove some bright or dark objects, and simplify the grey-levels of remaining objects, but they will not deform the contours between objects. They are thus very interesting image filters.

The extension of morphology on sets that we have described, is called *flat morphology*. This terminology arises from the fact that we work on the "horizontal" structure of functions (see Fig. 14.6). We will now see morphological operators on functions that act both "horizontally" and "vertically" on them.

As the operators will combine grey-levels by arithmetical additions and subtractions, it will no longer be possible to take a bounded interval for the grey-level set T, otherwise the grey-levels resulting from these operations might overflow out of this interval. Thus T must extend from $-\infty$ to $+\infty$. Let $T' = T \setminus \{-\infty, +\infty\}$; formally we have the following two requirements:

- T is closed under the operations of non-empty numerical supremum and infimum (thus T is a *complete lattice*);
- T' is closed under the operations of addition and subtraction (in other words, T' is a *subgroup* of $\mathbb{R}$).

It is then easily seen that either $T = \overline{\mathbb{R}}$ and $T' = \mathbb{R}$, or there is some $a > 0$ such that $T' = a\mathbb{Z} = \{az \mid z \in \mathbb{Z}\}$ and $T = \overline{a\mathbb{Z}} = a\mathbb{Z} \cup \{-\infty, +\infty\}$; in the second case, we can make a scaling of grey-levels by $1/a$, so here we can suppose without loss of generality that $T = \overline{\mathbb{Z}}$ and $T' = \mathbb{Z}$.

We gave above grey-level analogues of some set-theoretical operations. We have to extend this analogy further. First we redefine the *umbra* or *hypograph* of a function $F : E \to T$, it is the set

$$U(F) = \{(p, t) \in E \times T' \mid t \leq F(p)\} \ .$$

The difference with the previous definition is that we restrict t to T', while before we had $t \in T$. The points (p,t) of the umbra $U(F)$ are the analogues of the points $x \in X$ for a set X. We have now to give the analogue of a singleton, namely a set $\{p\}$ verifying $\{p\} \subseteq X \Leftrightarrow p \in X$; it is the *impulse* $i_{p,t}$, for $(p,t) \in E \times T'$, defined as follows:

$$\forall x \in E, \qquad i_{p,t}(x) = \begin{cases} t & \text{if } x = p, \\ -\infty & \text{if } x \neq p. \end{cases}$$

We verify indeed that for a function F and an impulse $i_{(p,t)}$, we have $i_{p,t} \leq F \Leftrightarrow (p,t) \in U(F)$.

We call the *support* of a function F the set

$$\operatorname{supp}(F) = \{p \in E \mid F(p) > -\infty\} \ .$$

Note that $p \in \operatorname{supp}(F)$ iff there exists some $t \in T'$ with $(p,t) \in U(F)$. We will see below that points outside the support are redundant in calculations; in fact, we can assume that F is defined only on its support; conversely if F is defined only on a subset S of E, we extend it to a function on E by setting $F(p) = -\infty$ for all $p \in E \setminus S$.

We defined above the translation of a function by a point. We extend it to the translation by a pair (p,t). Given a function $F : E \to T$ and a pair $(p,t) \in E \times T'$, the *translate of* F *by* (p,t) is the function $F_{(p,t)}$ whose graph is obtained by translating the graph $\{(x, F(x)) \mid x \in E\}$ by p in the first coordinate and by t in the second, that is,

$$\{(y, F_{(p,t)}(y)) \mid p \in E\} = \{(x+p, F(x)+t) \mid x \in E\} \ ,$$

in other words

$$\forall y \in E, \qquad F_{(p,t)}(y) = F(y-p) + t \ .$$

We can now define the Minkowski addition and subtraction of two functions $E \to T$, by analogy with Equation (14.1). Such an analogy already appeared partially in the definition of the dilation and erosion of a function by a set, Equation (14.10), but we have to extend it further. Given two functions $F, G : E \to T$, we define their Minkowski addition $F \oplus G$ and subtraction $F \ominus G$ as follows:

$$\begin{aligned} F \oplus G &= \bigvee_{(p,t) \in U(G)} F_{(p,t)} \ , \\ &= \bigvee_{(p,t) \in U(F)} G_{(p,t)} \ , \\ &= \bigvee \{ i_{(p+p',t+t')} \mid (p,t) \in U(F), (p',t') \in U(G)\} \ ; \\ F \ominus G &= \bigwedge_{(p,t) \in U(G)} F_{(-p,-t)} \ , \\ &= \bigvee \{ i_{(p,t)} \mid (p,t) \in E \times T', \ G_{(p,t)} \leq F \} \ . \end{aligned} \tag{14.12}$$

These two operations are illustrated in Fig. 14.7. Usually, F plays the role of a grey-level image, while G is the grey-level analogue of a structuring element, and we call it then a *structuring function*.

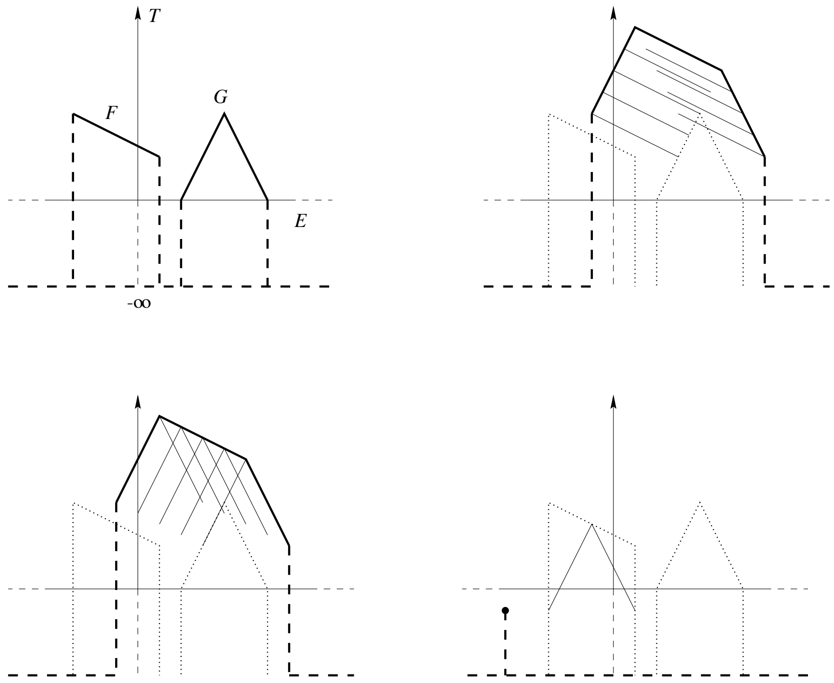


Figure 14.7. Top left: The two grey-level functions F and G, both having value $-\infty$ outside a bounded support. (In the next three illustrations, F and G are shown dotted.) Top right: $F \oplus G$ is the supremum of translates of F by all points of $U(G)$. Bottom left: It is also the supremum of translates of G by all points of $U(F)$. Bottom right: $F \ominus G$ is the supremum of all impulses $i_{(p,t)}$ such that $G_{(p,t)} \leq F$; in fact there is a unique point p for which this is possible, so $F \ominus G$ is an impulse.

We can give a numerical expression for the values of $F \oplus G$ and $F \ominus G$; for all $p \in E$ we have

$$\begin{aligned} (F \oplus G)(p) &= \sup_{h \in E} \big(F(p-h) + G(h)\big) \\ &= \sup_{h \in \mathrm{supp}(G)} \big(F(p-h) + G(h)\big) \ , \\ (F \ominus G)(p) &= \inf_{h \in E} \big(F(p+h) - G(h)\big) \\ &= \inf_{h \in \mathrm{supp}(G)} \big(F(p+h) - G(h)\big) \ , \end{aligned} \tag{14.13}$$

with the following convention for dealing with expressions of the form $+\infty - \infty$ inside the parentheses: if $F(p-h) + G(h)$ takes the form $+\infty - \infty$, we set it equal to $-\infty$, while if $F(p+h) - G(h)$ takes the form $+\infty - \infty$, we set it equal to $+\infty$.

The operators $\delta_G : T^E \to T^E : F \mapsto F \oplus G$ and $\varepsilon_G : T^E \to T^E : F \mapsto F \ominus G$ are called dilation and erosion by G. We can now define the binary operations $\circ$ and $\bullet$ as for sets, by $F \circ G = (F \ominus G) \oplus G$ and $F \bullet G = (F \oplus G) \ominus G$, leading thus to the opening by G, $\gamma_G : T^E \to T^E : F \mapsto F \circ G$, and the closing by G, $\varphi_G : T^E \to T^E : F \mapsto F \bullet G$; note that

$$F \circ G = \bigvee \{G_{(p,t)} \mid (p,t) \in E \times T',\ G_{(p,t)} \leq F\} \ ,$$

which is analogous to the second line in Equation (14.4). We still have the duality by inversion. Define the *transpose* or *symmetrical* $\widetilde{G}$ of G by $\widetilde{G}(x) = G(-x)$; we have the grey-level inversion N on functions (given by $N(F)(p) = -F(p)$); we get then

$$\text{(14.14)} \qquad \begin{array}{rclrcl} N\big(F \oplus G\big) & = & N(F) \ominus \widetilde{G} \ ; & N\big(F \ominus G\big) & = & N(F) \oplus \widetilde{G} \ ; \\ N\big(F \circ G\big) & = & N(F) \bullet \widetilde{G} \ ; & N\big(F \bullet G\big) & = & N(F) \circ \widetilde{G} \ . \end{array}$$

All properties of these operations $\oplus$, $\ominus$, $\circ$ and $\bullet$ in the case of sets, extend to the case of functions. For example, the opening and closing by G are idempotent. Operators on functions $E \to T$ built from these operations, together with the supremum and infimum, constitute what is called *grey-level morphology* or *functional morphology*.

Note finally that the dilation and erosion of functions by a set structuring element, Eqs. (14.10,14.11), are a particular case of dilation and erosion by a structuring function, Eqs. (14.12,14.13). Given a set $B \subseteq E$, define the function $B_0 : E \to T$ having value 0 on B, and $-\infty$ elsewhere:

$$\forall x \in E, \qquad B_0(x) = \begin{cases} 0 & \text{if } x \in B, \\ -\infty & \text{if } x \notin B. \end{cases}$$

Then for every function $F : E \to T$, we have $F \oplus B = F \oplus B_0$ and $F \ominus B = F \ominus B_0$. The function B_0 is thus called a *flat structuring function*. Hence flat morphology is a particular case of grey-level morphology, with a restriction of structuring functions to flat ones.

We explained above that flat operators behave on bright and dark parts of a grey-level image in the same way as the corresponding set operators do on foreground and background. This remains true here, but now the action is not only on the shape of these parts, but also on their grey-level profiles. For example, dilation and opening deform the grey-level profile on peaks, while erosion and closing do it on valleys. This is illustrated in Fig. 14.8; we see that the opening removes narrow peaks and the closing removes narrow valleys (as expected), but also the slope of jumps is reduced at the top with the opening, and at the bottom with the closing.

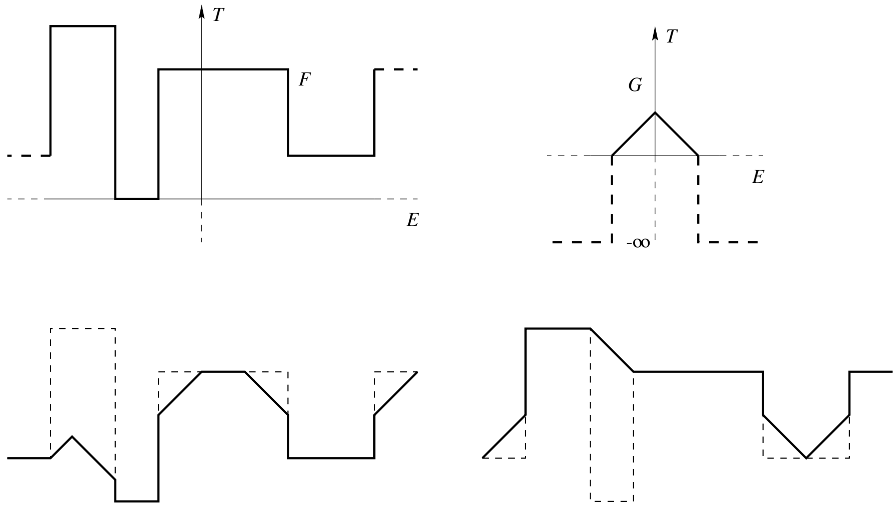


Figure 14.8. Top left: The grey-level function F. Top right: The structuring function G. Bottom left: $F \circ G$, the opening of F (dashed) by G. Bottom right: $F \bullet G$, the closing of F (dashed) by G.

For most practical problems concerning grey-level images, flat morphological operators are applied, instead of functional ones. Indeed, their expression is simpler (as it does not involve adding or subtracting grey-levels), and they work correctly for bounded grey-levels (functional ones can lead to overflow). In fact, flat operators have the same potential as functional ones for dealing with *spatial shapes* of objects in a grey-level image. However, there are sometimes situations where the grey-level profile of objects matters as much as their shape, and in such situations one will use functional morphological operators.

Let us say a few words about the computational complexity of morphological operations. Without any optimization, the complexity of the Minkowski operations is in $O(N \times S)$, where N is the size of the image and S is the size of the structuring element. However, thanks to various approaches, such as the decomposition of structuring elements, or the use of redundancies, it is possible for some particular types of structuring elements (say, rectangles), to have a complexity in $O(N \times \sqrt{S})$, $O(N \times \log S)$, or even $O(N)$. In digital grids using the usual connectivities based on neighbourhoods, geodesical reconstruction has a complexity in $O(N)$, thanks to the use of queues. In the binary case, pixels are inserted in the queue as soon as they receive a connected component label, and leave the queue when they transmit the label to their neighbours. For grey-level reconstruction, one uses a set of queues, one for each grey-level, with a priority order corresponding to the grey-level (e.g., for reconstruction by dilation, priority is given to the highest grey-levels).

2. Algebra

We saw in the Introduction how to define morphological operations on sets by combinations of unions, intersections and translations, and how these operations can be adapted to numerical functions by translating union and intersection into supremum and infimum. For many practical applications, such a framework resting on the analogy between sets and numerical functions, where foreground and background correspond to bright and dark image areas, is sufficient. However, if one wants to deepen the understanding of morphology, two questions come forward:

- Instead of extending the morphology on sets to the one on functions "by analogy", is there not a systematic approach that would give both as particular cases? Indeed, the early studies of grey-level morphology analysed the latter in terms of umbras of functions, they even attempted to make grey-level morphology a particular case of set morphology applied to umbras; however the correspondence between operations on functions and those on umbras is exact only for discrete grey-levels (Ronse, 1990).
- Can we define similarly morphological operations on other types of objects? For example, on the family $\mathcal{F}(\mathbb{R}^n)$ of closed subsets of $\mathbb{R}^n$: here an intersection of closed sets is closed, while a union of closed sets is not closed, but one could take instead the closure of their union; can we adapt Minkowski addition and subtraction in order to obtain all other morphological operators?

The answer to both questions is yes. Morphology on sets, on functions, and on several other types of objects (closed sets, convex sets, etc.) can be seen as particular cases of a general framework based on *complete lattices*. This was first introduced by Serra, 1988, then developed by Heijmans and Ronse, 1990, Ronse and Heijmans, 1991, Heijmans, 1991 and Heijmans, 1994. In this section, we present the algebraic fundamentals of mathematical morphology.

2.1 Complete lattice framework for images and operators

The basic idea is to generalize the notions of inclusion, union and intersection of sets, to other objects.

DEFINITION 14.1 *A* partial order *is a relation $\leq$ that is reflexive, anti-symmetrical and transitive. Write $\geq$ for the inverse of $\leq$ ($x \geq y$ iff $y \leq x$), it is also a partial order. A* partially ordered set *or* poset *is a pair $(X, \leq)$, where X is a set and $\leq$ a partial order on X.*

A complete lattice *is a poset $(X, \leq)$ in which every non-void part Y of X has a* least upper bound *or* supremum *$\bigvee Y$, and a* greatest lower bound *or* infimum *$\bigwedge Y$.*

It follows in particular that a complete lattice $(X, \leq)$ has always a *greatest element*, namely $\bigvee X$, and a *least element*, namely $\bigwedge X$. By analogy with Boolean algebras, the greatest (resp., least) element is also called the *one* (resp., *zero*), and it is written $\mathbf{1}$ or $\top$ (resp., $\mathbf{0}$ or $\perp$). Note also that every $x \in X$ is both lower bound and upper bound of the empty set. Hence:

$$\mathbf{1} = \bigvee X = \bigwedge \emptyset \quad \text{and} \quad \mathbf{0} = \bigwedge X = \bigvee \emptyset \ .$$

A *complete sublattice* of X is given by a subset Y of X, such that with the restriction to Y of the order $\leq$ on X, $(Y, \leq)$ is a *complete lattice* in which the supremum and infimum operations, as well as the zero and one, are identical to those in X; equivalently, it is a subset Y of X such that for every $Z \subseteq Y$, $\bigvee Z, \bigwedge Z \in Y$ (also for $Z = \emptyset$, i.e., $\mathbf{0}, \mathbf{1} \in Y$).

Some examples of complete lattices are particularly useful for mathematical morphology:

- The power set $\mathcal{P}(E)$, ordered by the set inclusion; here the supremum and infimum are the union and intersection. It represents the family of binary images.
- The grey-level sets $\overline{\mathbb{R}}$, $\overline{\mathbb{Z}}$, $[a, b]$ and $[a \ldots b]$ considered in Sec. 1.2 are complete lattices, and $\overline{\mathbb{Z}}$ is a complete sublattice of $\overline{\mathbb{R}}$.
- Given T one of the above complete lattices, and a space E, consider the set T^E of numerical functions $E \to T$. It is a complete lattice, in fact a *power lattice* of T, in the sense that its ordering, supremum and infimum derive from those on T by pointwise application, see Eqs. (14.7,14.8). It represents the family of grey-level images.
- The family $\mathcal{F}(\mathbb{R}^n)$ of closed subsets of $\mathbb{R}^n$ is a complete lattice for the ordering by inclusion; here the infimum of a family of closed sets is its intersection, while its supremum is the closure of its union. Despite the same ordering as in $\mathcal{P}(E)$, it is not a complete sublattice of $\mathcal{P}(E)$, because the supremum operation is not the same. Many metrics and topologies on sets are defined properly only for closed sets (Ronse and Tajine, 2004).
- We can represent RGB colours as triples (r, g, b) of numerical values, so T^3 is the complete lattice of RGB colours, with componentwise ordering; now we represent a RGB colour image as a function $E \to T^3$ associating to each point $p \in E$ a triple $\big(r(p), g(p), b(p)\big)$ coding the RGB colour of p; thus the family of RGB colour images constitutes the complete lattice $(T^3)^E$, with the componentwise ordering.
- Given a set E, the set $\Pi(E)$ of partitions on E is ordered as follows: given two partitions π_1 and π_2, we write $\pi_1 \leq \pi_2$ is π_1 is *finer* than

π_2, or equivalently π_2 is *coarser* than π_1; this means that every class of π_1 is included in a class of π_2. Then $\Pi(E)$ is a complete lattice. In fact, there is a one-to-one correspondence between partitions on E and equivalence relations on E; then the lattice structure of $\Pi(E)$ corresponds by bijection to the one of the family $Equiv(E)$ of equivalence relations on E, considered as a subset of E^2: the ordering on partitions corresponds to the inclusion order between equivalences, and the infimum and supremum of a family of partitions correspond respectively to the intersection and to the transitive closure of the union, of the associated equivalence relations.

Image processing operations can then be viewed as mappings $L \to L$, where L is the complete lattice of images under consideration; we can also consider mappings from one complete lattice to another, for example, $T^E \to \mathcal{P}(E)$ (binarization of grey-level images), or $T^E \to \Pi(E)$ (segmentation). Such mappings are usually written by Greek letters, and are called *operators*; an operator is said *on* L when it is a mapping $L \to L$. Given a set L (which can be a complete lattice or not) and a complete lattice M, the set M^L of operators $L \to M$ is a complete lattice, which inherits the order and complete lattice structure of M "componentwise", as happened for functions, see Eqs. (14.7,14.8): for two operators $\eta, \zeta : L \to M$, we have

$$\eta \leq \zeta \qquad \Longleftrightarrow \qquad \forall x \in L, \quad \eta(x) \leq \zeta(x) \ ,$$

and for a family ψ_i ($i \in I$) of operators $L \to M$, their supremum and infimum are given by:

$$\bigvee_{i \in I} \psi_i : L \to M : x \mapsto \bigvee_{i \in I} \psi_i(x) \quad \text{and} \quad \bigwedge_{i \in I} \psi_i : L \to M : x \mapsto \bigwedge_{i \in I} \psi_i(x) \ .$$

There is another operation on operators, *composition*; given $\eta : L \to M$ and $\zeta : M \to N$, the *composition of* η *followed by* ζ is the operator $\zeta\eta : L \to N : x \mapsto \zeta(\eta(x))$. Of particular interest is the composition of operators on L: the composition of two operators $\zeta, \eta : L \to L$ is always defined, and this gives an associative operation having as neutral element the *identity* $\mathbf{id} : L \to L : x \mapsto x$, in other words the set L^L of operators on L is what one calls a *monoid*. Given an operator ψ on L, we define recursively the power ψ^n for every $n \in \mathbb{N}$: $\psi^0 = \mathbf{id}$, $\psi^{n+1} = \psi[\psi^n]$. Let us recall some morphological terminology (Serra, 1988; Heijmans, 1994):

DEFINITION 14.2 *Given two posets L and M, an operator $\psi : L \to M$ is*

- increasing *(or* isotone*, Birkhoff, 1995) if for all $x, y \in L$, we have $x \leq y \ \Rightarrow \ \psi(x) \leq \psi(y)$.*
- decreasing *(or* antitone*, Birkhoff, 1995) if for all $x, y \in L$, we have $x \leq y \ \Rightarrow \ \psi(x) \geq \psi(y)$.*

- *an* isomorphism *if ψ is an increasing bijection, whose inverse ψ^{-1} is increasing.*
- *a* dual isomorphism *if ψ is a decreasing bijection, whose inverse ψ^{-1} is decreasing.*

Given a poset L, an operator ψ on L is

- extensive *if $\psi \geq \mathbf{id}$, that is, for every $x \in L$ we have $\psi(x) \geq x$.*
- anti-extensive *if $\psi \leq \mathbf{id}$, that is, for every $x \in L$ we have $\psi(x) \leq x$.*
- *an* automorphism of L *if ψ is an isomorphism $L \to L$.*
- *a* dual automorphism of L *if ψ is a dual isomorphism $L \to L$.*

Given a set L, an operator ψ on L is idempotent *if $\psi\psi = \psi$, that is, for every $x \in L$ we have $\psi(\psi(x)) = \psi(x)$.*

Note that if L is a complete lattice and ψ is an increasing operator on L, then we have (Heijmans and Ronse, 1990):

$$\forall (x_i, i \in I) \subseteq L, \qquad \begin{cases} \psi\Big(\bigvee_{i\in I} x_i\Big) & \geq & \bigvee_{i\in I} \psi(x_i) \ , \\ \psi\Big(\bigwedge_{i\in I} x_i\Big) & \leq & \bigwedge_{i\in I} \psi(x_i) \ . \end{cases} \tag{14.15}$$

Some other properties and specific families of operators (in particular, dilations, erosions, openings and closings) will be defined in the following subsections.

Given an operator ψ on a set L, the *invariance domain of* ψ is the set $Inv(\psi) = \{x \in L \mid \psi(x) = x\}$. Given an operator $\psi : L \to M$, the *range* (or *image*) of ψ is the set of $\psi(x)$ for $x \in L$; we write it $\psi(L)$.

In the lattice $\mathcal{P}(E)$ of parts of a Euclidean or digital space $E = \mathbb{R}^n$ or $\mathbb{Z}^n$, the dilation, erosion, opening and closing by a structuring element, Eqs. (14.1,14.4), are *translation-invariant*, in other words they commute with any translation of E. We can generalize this notion as follows. Let $\mathbf{T}$ be a *group* of automorphisms of the complete lattice L; in other words for every $\tau \in \mathbf{T}$, τ is an automorphism of L and $\tau^{-1} \in \mathbf{T}$, and for every $\tau_1, \tau_2 \in \mathbf{T}$, $\tau_1\tau_2 \in \mathbf{T}$. An operator ψ on L is said to be $\mathbf{T}$-*invariant* if it commutes with every element of $\mathbf{T}$: $\forall \tau \in \mathbf{T}$, $\tau\psi = \psi\tau$.

There is an important principle: *duality*. We saw above that the inverse $\geq$ of a partial order $\leq$ is a partial order. Therefore every notion concerning posets and complete lattices admits a *dual*, which is the same notion expressed w.r.t. the inverse order $\geq$; as the inverse of $\geq$ is again $\leq$, the duality is symmetrical. For example, the dual of the supremum operation is the infimum operation (and vice versa); for an operator, being extensive and being anti-extensive are dual properties. Note that all notions relying only on composition of operators, and not on order, are *auto-dual*; this is for example the case for the identity operator and for the property of idempotence.

An *inversion* of a poset L is a dual automorphism of L which is its own inverse, in other words a decreasing operator ν on L such that $\nu^2 = \mathbf{id}$. Then every operator ψ on L has a *dual by inversion*, $\psi^* = \nu\psi\nu$, whose properties are dual to those of ψ. For example, the set complementation in $\mathcal{P}(E)$, and the grey-level inversion (image negative) N in T^E, are inversions; in $\mathcal{P}(E)$ the dilation (resp., opening) by B is the dual by complementation of the erosion (resp., closing) by $\check{B}$, Eqs. (14.2,14.5), and in T^E the dilation and opening by G are the duals by inversion of the erosion and closing by $\widetilde{G}$, Equation (14.14).

2.2 Moore families, algebraic closings and openings

There are many mathematical situations where an object is "closed" under some operation: a closed set in a topological space, a convex set in $\mathbb{R}^n$, a subgroup of a group, a transitive relation. The interesting thing is that when an object is not closed, one can close it in a unique smallest possible way. From the algebraic point of view, it is thus fundamental to describe both the structure of the family of closed sets, and the properties of the closure operator.

DEFINITION 14.3 *Let L be a poset.*

1 *A subset M of L is a* Moore family *if every element of L has a least upper bound in M:*

$$\forall x \in L,\ \Big(\exists y \in M, y \geq x \textit{ and } \big[\forall z \in M,\ (z \geq x \ \Rightarrow\ z \geq y)\big]\Big)\ .$$

2 *A* closing *(or* closure operator*) on L is an increasing, extensive and idempotent operator $L \to L$.*

The Moore family stands for the family of closed objects. The equivalence between the two concepts of closed object and closing an object, is expressed as follows:

PROPOSITION 14.4 *Let L be a poset. There is a one-to-one correspondence between Moore families in L and closings on L, given as follows:*

- *To a Moore family M we associate the closing φ defined by setting for every $x \in L$: $\varphi(x)$ is equal to the least $y \in M$ such that $y \geq x$.*
- *To a closing φ one associates the Moore family M which is the* invariance domain *of φ: $M = Inv(\varphi)$.*

Note that $M = \{\varphi(x) \mid x \in L\}$. Let us now consider the case where L is a complete lattice.

THEOREM 14.5 *Let L be a complete lattice. A subset M of L is a Moore family iff M is closed under the infimum operation:*

$$\forall S \subseteq M, \ \bigwedge S \in M \ .$$

In particular, $\bigwedge \emptyset = \mathbf{1} \in M$. Given a Moore family M corresponding to a closing φ, $(M, \leq)$ is a complete lattice with greatest element $\mathbf{1}$ and least element $\varphi(\mathbf{0}) = \bigwedge M$, and where the supremum and infimum of a family $N \subseteq M$ are given by $\varphi\big(\bigvee N\big)$ and $\bigwedge N$, respectively.

Note that $\varphi(\mathbf{1}) = \mathbf{1}$ and $\varphi\big(\bigwedge N\big) = \bigwedge N$. Let us mention also the following property:

$$\text{(14.16)} \qquad \forall X \subseteq L, \quad \varphi\Big(\bigvee_{x \in X} \varphi(x)\Big) = \varphi\Big(\bigvee X\Big) \ .$$

Let us illustrate the above results with the family $\mathcal{F}$ of closed sets in a topological space E. Clearly $\mathcal{F}$ is a Moore family of $\mathcal{P}(E)$ (ordered by inclusion), which means that $\mathcal{F}$ is closed under arbitrary intersections, and contains the empty intersection $\bigcap \emptyset = E$; now $\mathcal{F}$ corresponds to a closing, which is the topological closure operator $\mathbf{cl}$, where for $X \subseteq E$, $\mathbf{cl}(X)$ is the least element of $\mathcal{F}$ containing X. However the Moore family $\mathcal{F}$ has two further properties:

1 $\emptyset \in \mathcal{F}$; by Theorem 14.5, this is equivalent to $\mathbf{cl}(\emptyset) = \emptyset$.

2 $\mathcal{F}$ is closed under binary union: for $C_1, C_2 \in \mathcal{F}$, $C_1 \cup C_2 \in \mathcal{F}$. By Theorem 14.5, this means that $\mathbf{cl}(C_1 \cup C_2) = C_1 \cup C_2$. Now $\mathcal{F}$ is the set of $\mathbf{cl}(X)$ for $X \in \mathcal{P}(E)$, and we can write $C_i = \mathbf{cl}(X_i)$, so in view of Equation (14.16), the condition is equivalent to:

$$\forall X_1, X_2 \in \mathcal{P}(E), \qquad \mathbf{cl}(X_1 \cup X_2) = \mathbf{cl}(X_1) \cup \mathbf{cl}(X_2) \ .$$

Therefore one can characterize a topology, given by the family of closed sets, through the associated closure operator $\mathbf{cl}$, which must be a closing (increasing, idempotent and extensive), preserve the empty set, and distribute binary union (Everett, 1944).

Let us now consider the dual concepts and results:

- In a poset L, a *dual Moore family* is a subset M such that every element of L has a greatest lower bound in M.
- The dual of a closing is an *opening* on L: an increasing, anti-extensive and idempotent operator $L \to L$.
- There is a one-to-one correspondence between dual Moore families in L and openings on L, where the corresponding opening γ and dual Moore

family M verify: $\gamma(x)$ is equal to the greatest $y \in M$ such that $y \leq x$, and M is the invariance domain of γ.

- In a complete lattice L, M is a dual Moore family iff M is closed under the supremum operation; in particular $\mathbf{0} \in M$. Given a dual Moore family M corresponding to an opening γ, $(M, \leq)$ is a complete lattice with greatest element $\gamma(\mathbf{1}) = \bigvee M$ and least element $\mathbf{0}$, and where the supremum and infimum of a family $N \subseteq M$ are given by $\bigvee N$ and $\gamma\big(\bigwedge N\big)$, respectively.
- A topology on a space E can be characterized by its topological interior operation $\mathbf{int}$, which is an opening verifying $\mathbf{int}(E) = E$ and $\mathbf{int}(X_1 \cap X_2) = \mathbf{int}(X_1) \cap \mathbf{int}(X_2)$.

Let us now describe the structure of the families of openings and closings. This will lead to some standard methods to construct them.

PROPOSITION 14.6 *Let L be a complete lattice.*

1. *The supremum of any family of openings on L is an opening, and the set of openings on L is a dual Moore family in L^L. For every increasing operator ψ on L, the greatest opening $\leq \psi$ is $\Gamma(\psi)$, it verifies $Inv(\Gamma(\psi)) = Inv(\mathbf{id} \wedge \psi)$.*
2. *The infimum of any family of closings is a closing, and the set of closings on L is a Moore family in L^L. For every increasing operator ψ on L, the least closing $\geq \psi$ is $\Phi(\psi)$, it verifies $Inv(\Phi(\psi)) = Inv(\mathbf{id} \vee \psi)$.*

By Proposition 14.4 (and its dual), for any $x \in L$, $\Gamma(\psi)(x)$ is the greatest $y \in Inv(\mathbf{id} \wedge \psi)$ such that $y \leq x$, and $\Phi(\psi)(x)$ is the least $y \in Inv(\mathbf{id} \vee \psi)$ such that $y \geq x$.

By Theorem 14.5 (and its dual), the set of openings (resp., closings) is a complete lattice, where $\mathbf{id}$ is the greatest opening (resp., the least closing), and the least opening is the constant operator $L \to L : x \mapsto \mathbf{0}$ (resp., the greatest closing is the constant operator $L \to L : x \mapsto \mathbf{1}$).

One can construct openings and closings by specifying some of their invariants. Let $b \in L$ and let $\mathbf{T}$ be a group of automorphisms of L. The *structural opening* and *closing* $\gamma_{b,\mathbf{T}}$ and $\varphi_{b,\mathbf{T}}$ are defined by

$$\forall x \in L, \quad \begin{cases} \gamma_{b,\mathbf{T}}(x) = \bigvee\{\tau(b) \mid \tau \in \mathbf{T},\ \tau(b) \leq x\} \ , \\ \varphi_{b,\mathbf{T}}(x) = \bigwedge\{\tau(b) \mid \tau \in \mathbf{T},\ \tau(b) \geq x\} \ . \end{cases} \tag{14.17}$$

More generally, given a family $S \subseteq L$, we define then

$$\gamma_{S,\mathbf{T}} = \bigvee_{s \in S} \gamma_{s,\mathbf{T}} \quad \text{and} \quad \varphi_{S,\mathbf{T}} = \bigwedge_{s \in S} \varphi_{s,\mathbf{T}} \ , \tag{14.18}$$

and we have

$$(14.19) \quad \forall x \in L, \quad \left\{ \begin{array}{l} \gamma_{S,\mathbf{T}}(x) = \bigvee \{\tau(s) \mid s \in S,\ \tau \in \mathbf{T},\ \tau(s) \leq x\} \ , \\ \varphi_{S,\mathbf{T}}(x) = \bigwedge \{\tau(s) \mid s \in S,\ \tau \in \mathbf{T},\ \tau(s) \geq x\} \ . \end{array} \right.$$

These operators are a $\mathbf{T}$-invariant opening and closing, respectively, and in fact every $\mathbf{T}$-invariant opening and closing takes this form:

PROPOSITION 14.7 *Let L be a complete lattice. For any $S \subseteq L$, let $\langle S \rangle_{\mathbf{T}}^{\sup}$ (resp., $\langle S \rangle_{\mathbf{T}}^{\inf}$) be the least subset of L containing S which is closed under $\mathbf{T}$ and under the supremum (resp., infimum) operation. We have*

$$\begin{array}{rrcl} & \langle S \rangle_{\mathbf{T}}^{\sup} & = & \left\{ \bigvee_{(\tau,s)\in X} \tau(s) \mid X \subseteq \mathbf{T} \times S \right\} \\ \textit{and} & \langle S \rangle_{\mathbf{T}}^{\inf} & = & \left\{ \bigwedge_{(\tau,s)\in X} \tau(s) \mid X \subseteq \mathbf{T} \times S \right\} . \end{array}$$

Then $\gamma_{S,\mathbf{T}}$ and $\varphi_{S,\mathbf{T}}$ are a $\mathbf{T}$-invariant opening and closing, respectively, with these sets as their respective invariance domain:

$$Inv(\gamma_{S,\mathbf{T}}) = \langle S \rangle_{\mathbf{T}}^{\sup} \quad \textit{and} \quad Inv(\varphi_{S,\mathbf{T}}) = \langle S \rangle_{\mathbf{T}}^{\inf} \ .$$

Conversely, every $\mathbf{T}$-invariant opening γ and closing φ take this form: $\gamma = \gamma_{Inv(\gamma),\mathbf{T}}$ and $\varphi = \varphi_{Inv(\varphi),\mathbf{T}}$.

A well-known example is when $L = \mathcal{P}(E)$, for $E = \mathbb{R}^n$ or $\mathbb{Z}^n$, and $\mathbf{T}$ is the group of translations of E. Then the structural opening and closing give the opening and closing by a structuring element: for every $X, B \in \mathcal{P}(E)$ we have $\gamma_{B,\mathbf{T}}(X) = X \circ B$ and $\varphi_{B,\mathbf{T}}(X) = (X^c \circ B^c)^c = X \bullet [\check{B}]^c$ (with $b \in [\check{B}]^c \Leftrightarrow -b \notin B$). For a family $\mathcal{S}$ of structuring elements, we get the openings and closings of the form given in Equation (14.6). We obtain thus the well-known fact that every translation-invariant opening (resp., closing) is a union of openings (resp., intersection of closings) by structuring elements.

When $\mathbf{T}$ reduces to the identity $\mathbf{id}$, we simply write γ_b, φ_b, γ_S and φ_S. Then the above result characterizes arbitrary openings and closings as being γ_S and φ_S for some $S \subseteq L$.

In the next subsection, we will see how openings and closings arise from dilations and erosions.

2.3 Galois connections and adjunctions

At the beginning of the 19th century, Evariste Galois built a connection between fields of numbers generated by roots of equations, and groups of permutations of these roots. This type of correspondence is the first example of a general technique used in algebra to build an association between two types of structures. It has thus been named after him.

DEFINITION 14.8 *Let A and B two posets, with two operators $\alpha : B \to A$ and $\beta : A \to B$. We say that α and β form a* Galois connection *if*

$$\forall a \in A,\ \forall b \in B, \qquad a \leq \alpha(b) \iff b \leq \beta(a)\ . \tag{14.20}$$

Note that α and β play symmetrical roles. Galois connections are often used in mathematical morphology to establish a dual isomorphism between two types of structures, thanks to the following result:

PROPOSITION 14.9 *Let A and B two posets, and let $\alpha : B \to A$ and $\beta : A \to B$ form a Galois connection. Then:*

1 *α and β are decreasing, $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$.*

2 *$\alpha\beta$ is a closing on A, $\beta\alpha$ is a closing on B, $Inv(\alpha\beta) = \alpha(B)$ and $Inv(\beta\alpha) = \beta(A)$ (so that $\alpha(B)$ and $\beta(A)$ are Moore families).*

3 *The restriction of β to $\alpha(B)$ is a dual isomorphism $\alpha(B) \to \beta(A)$ whose inverse $\beta(A) \to \alpha(B)$ is the restriction of α to $\beta(A)$.*

One can generally characterize the types of maps α and β which may appear in a Galois connection, but in the case of complete lattices, this characterization is straightforward:

DEFINITION 14.10 *Let A and B be complete lattices. An operator $\alpha : B \to A$ is a* Galois map *if it exchanges supremum and infimum:*

$$\forall (x_i, i \in I) \subseteq B, \qquad \alpha\Big(\bigvee_{i\in I} x_i\Big) = \bigwedge_{i \in I} \alpha(x_i)\ .$$

In particular (for $I = \emptyset$), α maps the least element $\mathbf{0}_B$ of B onto the greatest element $\mathbf{1}_A$ of A.

PROPOSITION 14.11 *Let A and B be complete lattices. Then:*

1 *Given $\alpha : B \to A$ and $\beta : A \to B$ forming a Galois connection, α and β are Galois maps.*

2 *Conversely, given a Galois map $\alpha : B \to A$, there is a unique Galois map $\beta : A \to B$ such that α and β form a Galois connection (and vice versa).*

3 *Given $\alpha_1, \alpha_2 : B \to A$ and $\beta_1, \beta_2 : A \to B$ such that α_i and β_i form a Galois connection for $i = 1, 2$, we have $\alpha_1 \leq \alpha_2 \Leftrightarrow \beta_1 \leq \beta_2$.*

4 *Given $\alpha_i : B \to A$ and $\beta_i : A \to B$ forming a Galois connection for $i \in I$, $\bigwedge_{i\in I} \alpha_i$ and $\bigwedge_{i\in I} \beta_i$ form a Galois connection.*

In other words, Galois maps form a Moore family in the complete lattice of operators $A \to B$ (or $B \to A$), and Galois connection establishes an isomorphism between the two complete lattices of Galois maps $A \to B$ and $B \to A$.

Of particular interest are Galois connections between subsets of two sets, which were characterized by Ore, 1944 in terms of a relation between the points of the two sets:

THEOREM 14.12 *Let V and W two sets.*

1 *Given a relation ρ between elements of V and of W, define*

$$\begin{aligned} \alpha_\rho : \quad & \mathcal{P}(W) \to \mathcal{P}(V) : \quad Y \mapsto \{v \in V \mid \forall w \in Y,\ v\ \rho\ w\}\ , \\ \beta_\rho : \quad & \mathcal{P}(V) \to \mathcal{P}(W) : \quad X \mapsto \{w \in W \mid \forall v \in X,\ v\ \rho\ w\}\ . \end{aligned}$$

Then α_ρ and β_ρ form a Galois connection.

2 *Conversely, given $\alpha : \mathcal{P}(W) \to \mathcal{P}(V)$ and $\beta : \mathcal{P}(V) \to \mathcal{P}(W)$ forming a Galois connection, there is a unique relation ρ between elements of V and of W, such that $\alpha = \alpha_\rho$ and $\beta = \beta_\rho$; the relation ρ is given by*

$$\forall v \in V, \forall w \in W, \qquad v\ \rho\ w \iff v \in \alpha(\{w\}) \iff w \in \beta(\{v\})\ .$$

Following Birkhoff, 1995, the Galois maps α_ρ and β_ρ are called *polarities*. Galois connections between sets expressed in such a form, arise in many aspects of mathematics and computer science. See for example Sec. 3.1.

We turn now to the notion of adjunction, which is "semi-dual" to the one of Galois connection, in the sense that we reverse the ordering on one of the posets, but not on the other.

DEFINITION 14.13 *Let A and B two posets, with two operators and $\delta : A \to B$ and $\varepsilon : B \to A$. We say that (ε, δ) is an* adjunction *if*

$$\forall a \in A,\ \forall b \in B, \qquad \delta(a) \le b \iff a \le \varepsilon(b)\ . \tag{14.21}$$

We say that δ is lower adjoint *of ε, and ε is* upper adjoint *of δ.*

Compared with Galois connections (see Definition 14.8), we have reversed the ordering on B, since we have $\delta(a) \le b$ instead of $b \le \delta(a)$. Hence ε and δ do not play symmetrical roles, that is why we write the ordered pair (ε, δ). We obtain then the analogue of Proposition 14.9:

PROPOSITION 14.14 *Let A and B two posets, and let $\delta : A \to B$ and $\varepsilon : B \to A$ such that (ε, δ) is an adjunction. Then:*

1 *ε and δ are increasing, $\varepsilon = \varepsilon\delta\varepsilon$ and $\delta = \delta\varepsilon\delta$.*

2 *$\varepsilon\delta$ is a closing on A, $\delta\varepsilon$ is an opening on B, $Inv(\varepsilon\delta) = \varepsilon(B)$ and $Inv(\delta\varepsilon) = \delta(A)$ (so that $\varepsilon(B)$ is a Moore family and $\delta(A)$ is a dual Moore family).*

3 *The restriction of δ to $\varepsilon(B)$ is an isomorphism $\varepsilon(B) \to \delta(A)$ whose inverse $\delta(A) \to \varepsilon(B)$ is the restriction of ε to $\delta(A)$.*

PROPOSITION 14.15 *Let L be a poset, $\mathbf{T}$ a group of automorphisms of L, and $\varepsilon, \delta : L \to L$ such that (ε, δ) is an adjunction. Then ε is $\mathbf{T}$-invariant iff δ is $\mathbf{T}$-invariant.*

Let us now characterize adjunctions in the case of complete lattices.

DEFINITION 14.16 *Let A and B be complete lattices.*

1 *An operator $\varepsilon : B \to A$ is an* erosion *if it commutes with the infimum operation:*

$$\forall (x_i, i \in I) \subseteq B, \qquad \varepsilon\Big(\bigwedge_{i \in I} x_i\Big) = \bigwedge_{i \in I} \varepsilon(x_i) \ .$$

In particular (for $I = \emptyset$), ε maps the greatest element $\mathbf{1}_B$ of B onto the greatest element $\mathbf{1}_A$ of A.

2 *An operator $\delta : B \to A$ is a* dilation *if it commutes with the supremum operation:*

$$\forall (x_i, i \in I) \subseteq B, \qquad \delta\Big(\bigvee_{i \in I} x_i\Big) = \bigvee_{i \in I} \delta(x_i) \ .$$

In particular (for $I = \emptyset$), δ maps the least element $\mathbf{0}_B$ of B onto the least element $\mathbf{0}_A$ of A.

Note that dilations and erosions are increasing. Also the set of $\delta(x)$ ($x \in B$) is closed under the supremum operation, while the set of $\varepsilon(x)$ ($x \in B$) is closed under the infimum operation. We obtain now the analogue of Proposition 14.11:

THEOREM 14.17 *Let A and B be complete lattices. Then:*

1 *Given $\delta : A \to B$ and $\varepsilon : B \to A$ such that (ε, δ) is an adjunction, δ is a dilation and ε is an erosion.*

2 *Conversely, (a) given a dilation $\delta : A \to B$, there is a unique erosion $\varepsilon : B \to A$ such that (ε, δ) is an adjunction, and*

(b) given an erosion $\varepsilon : B \to A$, there is a unique dilation $\delta : A \to B$ such that (ε, δ) is an adjunction.

3 *Given $\delta_1, \delta_2 : A \to B$ and $\varepsilon_1, \varepsilon_2 : B \to A$ such that $(\varepsilon_i, \delta_i)$ is an adjunction for $i = 1, 2$, we have $\delta_1 \leq \delta_2 \Leftrightarrow \varepsilon_1 \geq \varepsilon_2$.*

4 *Given $\delta_i : A \to B$ and $\varepsilon_i : B \to A$ such that $(\varepsilon_i, \delta_i)$ is an adjunction for $i \in I$, $\left(\bigwedge_{i \in I} \varepsilon_i, \bigvee_{i \in I} \delta_i\right)$ is an adjunction.*

In other words, in the complete lattice of operators $A \to B$ (or $B \to A$), erosions form a Moore family, while dilations form a dual Moore family, and adjunctions establish a dual isomorphism between the two complete lattices of dilations $A \to B$ and erosions $B \to A$.

The classical example of adjunction is given by the erosion and dilation by a structuring element or function, Eqs. (14.1,14.10,14.12), arising from the Minkowski addition and subtraction. They are both translation-invariant (cf. Proposition 14.15). Here $A = B = \mathcal{P}(E)$ or T^E. In fact, every translation invariant dilation/erosion on sets arises from Minkowski operations, Equation (14.1), while for functions, every flat dilation/erosion invariant under spatial translations takes the form of Equation (14.10), and every dilation/erosion invariant under both spatial and grey-level translations arises from Minkowski operations, Equation (14.12).

In Heijmans and Ronse, 1990, there is a general study of complete lattices where it is possible to define such Minkowski operations, and to obtain for them properties similar to those verified for sets. Particular cases include of courses $\mathcal{P}(E)$ and T^E ($E = \mathbb{R}^n$ or $\mathbb{Z}^n$, $T = \overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$), for which we obtain the form given in Eqs. (14.1,14.12), but also: the lattice of convex subsets of $\mathbb{R}^n$ (here the supremum is the convex hull of the union, but Minkowski operations are the same as in $\mathcal{P}(\mathbb{R}^n)$), the lattice $\mathcal{F}(\mathbb{R}^n)$ of closed sets of $\mathbb{R}^n$ (here the supremum is the closure of the union, and the Minkowski addition is the closure of the one obtained in $\mathcal{P}(\mathbb{R}^n)$, but the Minkowski subtraction is the one of $\mathcal{P}(\mathbb{R}^n)$), upper semi-continuous functions $R^n \to \overline{\mathbb{R}}$, etc.

In the case where $A = B$, the operators ε, δ are $A \to A$, and can be composed arbitrarily in any order. It is then easily checked that in a poset A we have

(14.22)
$$\delta \geq \mathbf{id} \Leftrightarrow \delta \geq \varepsilon\delta \Leftrightarrow \delta\varepsilon \geq \varepsilon \Leftrightarrow \mathbf{id} \geq \varepsilon$$

and

(14.23)
$$\delta^2\varepsilon \leq \mathbf{id} \Leftrightarrow \delta^2 \leq \delta \Leftrightarrow \delta \leq \varepsilon\delta \Leftrightarrow \delta\varepsilon \leq \varepsilon \Leftrightarrow \varepsilon \leq \varepsilon^2 \Leftrightarrow \mathbf{id} \leq \varepsilon^2\delta \ .$$

This gives then the following result, which will be used later on, in the case of sets:

PROPOSITION 14.18 *Let A be a poset, and let (ε, δ) be an adjunction (for $\delta, \varepsilon : A \to A$). Then the following five statement are equivalent: (a) δ is a closing, (b) ε is an opening, (c) $\delta\varepsilon = \varepsilon$, (d) $\varepsilon\delta = \delta$, (e) δ and ε verify one statement of Equation (14.22) and one statement of Equation (14.23). Then we have*

$$\begin{aligned} &Inv(\varepsilon\delta) = Inv(\delta\varepsilon) = Inv(\delta) = Inv(\varepsilon) \\ &= \varepsilon\delta(A) = \delta\varepsilon(A) = \delta(A) = \varepsilon(A) \ . \end{aligned}$$

This set is both a Moore family and a dual Moore family in A; when A is a complete lattice, it is a complete sublattice of A.

Let us now consider dilations, erosions and adjunctions on sets. Let V and W two sets, and let ρ be a relation between elements of V and of W. We define $\delta_\rho : \mathcal{P}(V) \to \mathcal{P}(W)$, the dilation by ρ, and $\varepsilon_\rho : \mathcal{P}(W) \to \mathcal{P}(V)$, the erosion by ρ, as follows:

$$\begin{aligned} &\forall X \in \mathcal{P}(V), \quad \delta_\rho(X) = \{w \in W \mid \exists v \in X,\ v\ \rho\ w\} \ , \\ &\forall Y \in \mathcal{P}(W), \quad \varepsilon_\rho(Y) = \{v \in V \mid \forall w \in W,\ v\ \rho\ w \Rightarrow w \in Y\} \ . \end{aligned} \tag{14.24}$$

Alternately, we can define dilation erosion in terms of a map $N : V \to \mathcal{P}(W)$ and the *dual* map $\widetilde{N} : W \to \mathcal{P}(V)$, corresponding to the relation ρ by

$$\forall v \in V, \forall w \in W, \quad \begin{cases} w \in N(v) \ \Leftrightarrow \ v \in \widetilde{N}(w) \ \Leftrightarrow \ v\ \rho\ w \ , \\ \text{that is,} \quad N(v) = \{w \in W \mid v\ \rho\ w\} \\ \quad\text{and} \quad \widetilde{N}(w) = \{v \in V \mid v\ \rho\ w\} \ . \end{cases} \tag{14.25}$$

When $V = W$, the set $N(v)$ can be considered as the window or neighbourhood of point v, and N is called a *neighbourhood function* or a *windowing function*. Now Equation (14.24) can be written

$$\begin{aligned} &\forall X \in \mathcal{P}(V), \quad \delta_N(X) = \bigcup_{v \in X} N(v) = \{w \in W \mid \widetilde{N}(w) \cap X \neq \emptyset\} \ , \\ &\forall Y \in \mathcal{P}(W), \quad \varepsilon_N(Y) = \{v \in V \mid N(v) \subseteq Y\} \ . \end{aligned} \tag{14.26}$$

We have then the analogue for adjunctions of Ore's characterization of Galois connections on sets (Theorem 14.12):

THEOREM 14.19 *Let V and W two sets.*

1. *Given a map $N : V \to \mathcal{P}(W)$, $(\varepsilon_N, \delta_N)$ is an adjunction.*

2. *Conversely, given $\delta : \mathcal{P}(V) \to \mathcal{P}(W)$ and $\varepsilon : \mathcal{P}(W) \to \mathcal{P}(V)$ such that (ε, δ) is an adjunction, there is a unique map $N : V \to \mathcal{P}(W)$ such that $\delta = \delta_N$ and $\varepsilon = \varepsilon_N$; for every $v \in V$, $N(v) = \delta(\{v\})$.*

Note that $\delta_{\widetilde{N}}$ is a dilation $\mathcal{P}(W) \to \mathcal{P}(V)$, $\varepsilon_{\widetilde{N}}$ is an erosion $\mathcal{P}(V) \to \mathcal{P}(W)$, $(\varepsilon_{\widetilde{N}}, \delta_{\widetilde{N}})$ is an adjunction, and that $\delta_{\widetilde{N}}$ and $\varepsilon_{\widetilde{N}}$ are dual by complementation of ε_N and δ_N respectively, as

$$\begin{array}{rll} & \forall Y \in \mathcal{P}(W), & \delta_{\widetilde{N}}(Y) = V \setminus \varepsilon_N(W \setminus Y) \\ \text{and} & \forall X \in \mathcal{P}(V), & \varepsilon_{\widetilde{N}}(X) = W \setminus \delta_N(V \setminus X) \ . \end{array}$$

In fact $\delta_{\widetilde{N}} = \delta_{\rho^{-1}}$ and $\varepsilon_{\widetilde{N}} = \varepsilon_{\rho^{-1}}$, where ρ^{-1} is the relation inverse of ρ ($w \, \rho^{-1} \, v \;\Leftrightarrow\; v \, \rho \, w$).

A classical example is given for $V = W = E$ for E being the Euclidean space $\mathbb{R}^n$ or the digital space $\mathbb{Z}^n$, and the neighbourhoods being built from a structuring element $B \subseteq E$: for every $p \in E$, $N(p) = B_p$. Then $\widetilde{N}(p) = (\check{B})_p$ for all $p \in E$, $\delta_N = \delta_B$, $\varepsilon_N = \varepsilon_B$, $\delta_{\widetilde{N}} = \delta_{\check{B}}$ and $\varepsilon_{\widetilde{N}} = \varepsilon_{\check{B}}$. These operators are translation-invariant. In fact, from Proposition 14.15, for an adjunction $(\varepsilon_N, \delta_N)$, ε_N is translation-invariant iff δ_N is translation-invariant, and in such a case it is easily seen that they are the erosion and dilation by the structuring element $B = N(o)$.

PROPOSITION 14.20 *The following are equivalent:*

$$\big(\forall v \in V,\ N(v) \neq \emptyset\big) \;\Leftrightarrow\; \varepsilon_N(\emptyset) = \emptyset \;\Leftrightarrow\; \delta_{\widetilde{N}}(W) = V \;\Leftrightarrow\; \varepsilon_N \leq \delta_{\widetilde{N}} \ .$$

Dually, the following are equivalent:

$$\big(\forall w \in W,\ \widetilde{N}(w) \neq \emptyset\big) \;\Leftrightarrow\; \varepsilon_{\widetilde{N}}(\emptyset) = \emptyset \;\Leftrightarrow\; \delta_N(V) = W \;\Leftrightarrow\; \varepsilon_{\widetilde{N}} \leq \delta_N \ .$$

This result will intervene later on, in particular in Sec. 3.2 and Sec. 4. Note that in the case where $V = W = E$ ($E = \mathbb{R}^n$ or $\mathbb{Z}^n$) and $N(p) = B_p$ for all $p \in E$, the two equivalences reduce both to $B \neq \emptyset$.

Consider now the case where $V = W = E$. Here ρ is a relation on E, and both N and $\widetilde{N}$ are $E \to \mathcal{P}(E)$. The following two results will be used in Sec. 3.2:

PROPOSITION 14.21 *Consider a relation ρ on a set E, and the corresponding maps $N, \widetilde{N} : E \to \mathcal{P}(E)$. Then:*

1. *The following five statements are equivalent: (a) ρ is reflexive, (b) δ_N is extensive, (c) ε_N is anti-extensive, (d) $\delta_{\widetilde{N}}$ is extensive, (e) $\varepsilon_{\widetilde{N}}$ is anti-extensive.*

2. *The following five statements are equivalent: (a) ρ is symmetrical, (b) $\varepsilon_{\widetilde{N}}\delta_N$ is extensive, (c) $\delta_N\varepsilon_{\widetilde{N}}$ is anti-extensive, (d) $\varepsilon_N\delta_{\widetilde{N}}$ is extensive, (e) $\delta_{\widetilde{N}}\varepsilon_N$ is anti-extensive.*

3. *The following five statements are equivalent: (a) ρ is transitive, (b) $\delta_N^2 \leq \delta_N$, (c) $\varepsilon_N^2 \geq \varepsilon_N$, (d) $\delta_{\widetilde{N}}^2 \leq \delta_{\widetilde{N}}$, (e) $\varepsilon_{\widetilde{N}}^2 \geq \varepsilon_{\widetilde{N}}$.*

Combining items 1 and 3 with Proposition 14.18, we deduce:

PROPOSITION 14.22 *Consider a relation ρ on E, and the corresponding maps $N, \widetilde{N} : E \to \mathcal{P}(E)$. Then the following nine statements are equivalent: (a) ρ is reflexive and transitive, (b) δ_N is a closing, (c) ε_N is an opening, (d) $\delta_N \varepsilon_N = \varepsilon_N$, (e) $\varepsilon_N \delta_N = \delta_N$, (f) $\delta_{\widetilde{N}}$ is a closing, (g) $\varepsilon_{\widetilde{N}}$ is an opening, (h) $\delta_{\widetilde{N}} \varepsilon_{\widetilde{N}} = \varepsilon_{\widetilde{N}}$, (i) $\varepsilon_{\widetilde{N}} \delta_{\widetilde{N}} = \delta_{\widetilde{N}}$. We have then*

$$\begin{aligned} & Inv(\varepsilon_N \delta_N) = Inv(\delta_N \varepsilon_N) = Inv(\delta_N) = Inv(\delta_N) \\ & = \{\varepsilon_N \delta_N(Z) \mid Z \in \mathcal{P}(E)\} = \{\delta_N \varepsilon_N(Z) \mid Z \in \mathcal{P}(E)\} \\ & = \{\delta_N(Z) \mid Z \in \mathcal{P}(E)\} = \{\varepsilon_N(Z) \mid Z \in \mathcal{P}(E)\} \ , \end{aligned}$$

and the same with $\widetilde{N}$ in place of N. The two families $Inv(\varepsilon_N \delta_N) = Inv(\delta_N \varepsilon_N)$ and $Inv(\varepsilon_{\widetilde{N}} \delta_{\widetilde{N}}) = Inv(\delta_{\widetilde{N}} \varepsilon_{\widetilde{N}})$ are closed under arbitrary union and intersection, and contain E and $\emptyset$ (in other words they are complete sublattices of $(\mathcal{P}(E), \subseteq)$).

An *Alexandroff topology* (Alexandroff, 1937; Alexandroff and Hopf, 1935) is a topological space $(E, \mathcal{G})$ where the family $\mathcal{G}$ of open sets is closed under arbitrary intersection; in other words $\mathcal{G}$ is a complete sublattice of $(\mathcal{P}(E), \subseteq)$. It is equivalent to require that every point of E has a least open neighbourhood. By the *Alexandroff specialization theorem* (Alexandroff, 1956), there is a one-to-one correspondence between Alexandroff topologies on E and reflexive and transitive relations on E; in fact, for $x, y \in E$, $x \ \rho \ y$ iff x is in the closure of $\{y\}$, i.e., iff y belongs to the least neighbourhood of x. It follows then that for $x \in E$, $N(x)$ is the least neighbourhood of x and $\widetilde{N}(x)$ is the topological closure of $\{x\}$, while for $X \in \mathcal{P}(E)$, $\delta_N(X)$ is the least open set containing X (called the *star* of X), $\varepsilon_N(X)$ is the topological interior of X, $\delta_{\widetilde{N}}(X)$ is the topological closure of X, and $\varepsilon_{\widetilde{N}}(X)$ is the greatest closed subset of X. Note that $Inv(\varepsilon_N \delta_N) = Inv(\delta_N \varepsilon_N)$ is the family of open sets and $Inv(\varepsilon_{\widetilde{N}} \delta_{\widetilde{N}}) = Inv(\delta_{\widetilde{N}} \varepsilon_{\widetilde{N}})$ is the family of closed sets.

We saw in Sec. 2.2 that a closing φ on $\mathcal{P}(E)$ is the closure operator in a topology on E iff it satisfies the following two additional constraints: $\varphi(\emptyset) = \emptyset$ and $\varphi(X_1 \cup X_2) = \varphi(X_1) \cup \varphi(X_2)$ for all $X_1, X_2 \in \mathcal{P}(E)$; we have then $\varphi(X_1 \cup \cdots \cup X_n) = \varphi(X_1) \cup \cdots \cup \varphi(X_n)$ for all $X_1, \ldots, X_n \in \mathcal{P}(E)$. In other words the commutation with the union operation, $\varphi\big(\bigcup_{i \in I} X_i\big) = \bigcup_{i \in I} \varphi(X_i)$, is verified for I being empty or finite. This is weaker than φ being a dilation, where this identity is verified also for an infinite family I; but then the set of closed sets $\varphi(X)$ is closed under infinite unions, which means indeed that we have an Alexandroff topology.

2.4 Morphological filters

The word "filter" is used in several scientific and technological contexts, with various meanings. In image processing, one knows the linear filters, namely convolution operators, in particular the bandpass filter from signal processing, which preserves all frequencies within a band, and eliminates all others. In non-linear image processing, the well-known median filter has been used to remove impulsive noise, without the blurring effect of linear smoothing filters. The morphological approach to filtering is similar to that of signal processing, namely preserving some parts of an image and eliminating some others, except that the separation of these parts is not based on frequencies. The model proposed is that of an *ideal filter*, i.e., one that keeps the wanted components unaltered, and eliminates completely the unwanted ones. In order to characterize an ideal filter, rather than describing the features to be preserved or removed, one takes an algebraic point of view: if the filter does not alter the wanted parts and eliminates completely the unwanted ones, then applying the filter a second time will not change anything. Hence the main characteristic of an ideal filter is its idempotence. This is important from a theoretical point of view, but also for practical applications: if after applying the filter on an image the result is not satisfying, then we know that another filter must be applied. This contrasts with the behaviour of the median filter: after one application, some noise remains, that could be eliminated by a second or third application; then one can repeat the application of the filter, without guarantee that this will lead to a stable final result, as the median filter can produce oscillations (Serra, 1988). This is related to the fact that one cannot characterize precisely what are the features preserved or eliminated by this filter.

Besides idempotence, mathematical morphology demands that the behaviour of a filter should be related to the order and complete lattice structure of the family of images. Therefore one calls a *morphological filter* (or simply, a *filter*) an increasing and idempotent operator on a poset (or complete lattice). Write $Filt(L)$ for the set of filters on L. We have already encountered some filters: openings and closings. There are many other ones, and we will describe here some techniques for constructing them. This requires some terminology:

DEFINITION 14.23 *Let L be a poset and ψ an operator on L. We say that:*

1 ψ *is* underpotent *if* $\psi^2 \leq \psi$.

2 ψ *is* overpotent *if* $\psi^2 \geq \psi$.

3 ψ *is an* underfilter *if* ψ *is increasing and underpotent.*

4 ψ *is an* overfilter *if* ψ *is increasing and overpotent.*

We saw in Proposition 14.6 that in a complete lattice, the set of openings is a dual Moore family and the set of closings is a Moore family. They constitute thus two complete lattices. We have a similar result for filters (Serra, 1988):

PROPOSITION 14.24 *Let L be a complete lattice.*

1 *The set of overfilters on L is a dual Moore family in L^L, i.e. it is closed under the supremum operation.*

2 *The set of underfilters on L is a Moore family in L^L, i.e. it is closed under the infimum operation.*

3 *The set $Filt(L)$ of filters on L is a complete lattice. For any family ψ_i ($i \in I$) of filters, their supremum in $Filt(L)$ is the least underfilter ψ such that $\psi \geq \bigvee_{i \in I} \psi_i$, and their infimum in $Filt(L)$ is the greatest overfilter ψ such that $\psi \leq \bigwedge_{i \in I} \psi_i$.*

This gives a first method for constructing a filter from a family of filters. The second one arises from composition (Serra, 1988):

PROPOSITION 14.25 *Let L be a complete lattice and let ξ and ψ be two filters on L such that $\xi \geq \psi$. Then:*

1 *The only operators that can be obtained by repeated compositions of ψ and ξ are $\psi\xi$, $\xi\psi$, $\psi\xi\psi$ and $\xi\psi\xi$. They are all filters and*

$$\xi \geq \xi\psi\xi \geq \left\{ \begin{array}{c} \psi\xi \\ \xi\psi \end{array} \right\} \geq \psi\xi\psi \geq \psi \ .$$

2 $Inv(\xi) \cap Inv(\psi) \subseteq Inv(\xi\psi\xi) = Inv(\xi\psi) \subseteq Inv(\xi)$ *and* $Inv(\xi) \cap Inv(\psi) \subseteq Inv(\psi\xi\psi) = Inv(\psi\xi) \subseteq Inv(\psi)$.

3 *In $Filt(L)$, the supremum and infimum of $\xi\psi$ and $\psi\xi$ are $\xi\psi\xi$ and $\psi\xi\psi$ respectively.*

Note that items 1 and 2 do not require L to be a complete lattice, they are valid in any poset. A classical example is when ξ is a closing and ψ is an opening: the opening filters out positive noise, the closing filters out negative noise, so the composition of the two should filter out both types of noise (cf. Sec. 1.1).

The above result is at the basis of a well-known filter introduced in the 1980s, the *alternate sequential filter*. Suppose that we have an image where features of foreground and background are imbricated. To extract an object of a given size, it is necessary to filter its holes at a smaller size, and this require filtering objects at an even smaller size, etc. Thus we will apply openings and closings at increasing scales in order to simplify the image. Consider n openings $\gamma_1, \ldots, \gamma_n$ such that $\gamma_n \leq \cdots \leq \gamma_1$, and n closings $\varphi_1, \ldots, \varphi_n$ such that $\varphi_1 \leq \cdots \leq \varphi_n$. From the previous proposition, the compositions $\mu_i = \gamma_i\varphi_i$, $\nu_i = \varphi_i\gamma_i$, $\rho_i = \varphi_i\gamma_i\varphi_i$ and $\sigma_i = \gamma_i\varphi_i\gamma_i$ are filters. Alternate sequential filters are then defined as:

$$\begin{array}{rcl} \mu_i\mu_{i-1}\cdots\mu_2\mu_1 & = & (\gamma_i\varphi_i)(\gamma_{i-1}\varphi_{i-1})\cdots(\gamma_2\varphi_2)(\gamma_1\varphi_1) \ , \\ \nu_i\nu_{i-1}\cdots\nu_2\nu_1 & = & (\varphi_i\gamma_i)(\varphi_{i-1}\gamma_{i-1})\cdots(\varphi_2\gamma_2)(\varphi_1\gamma_1) \ , \end{array} \tag{14.27}$$

or as the following variants:
(14.28)

$$\begin{aligned}\rho_i\rho_{i-1}\cdots\rho_2\rho_1 &= (\varphi_i\gamma_i\varphi_i)(\varphi_{i-1}\gamma_{i-1}\varphi_{i-1})\cdots(\varphi_2\gamma_2\varphi_2)(\varphi_1\gamma_1\varphi_1)\\ &= \varphi_i\mu_i\mu_{i-1}\cdots\mu_2\mu_1\ ,\\ \sigma_i\sigma_{i-1}\cdots\sigma_2\sigma_1 &= (\gamma_i\varphi_i\gamma_i)(\gamma_{i-1}\varphi_{i-1}\gamma_{i-1})\cdots(\gamma_2\varphi_2\gamma_2)(\gamma_1\varphi_1\gamma_1)\\ &= \gamma_i\nu_i\nu_{i-1}\cdots\nu_2\nu_1\ ,\end{aligned}$$

for $i = 1, \dots, n$. They are all filters. They are useful for filtering images where grains (bright zones) are imbricated with pores (dark zones) at all sizes. Typically, the γ_i's and φ_i's can be:

- openings and closings by structuring elements of increasing sizes;
- openings and closings by reconstruction, based on structuring elements of increasing sizes;
- area openings and closings (removing grains and pores on the basis of their area), with increasing area thresholds;

hence as i increases, the alternating sequential filters will progressively remove grains and pores of increasing sizes, thus simplifying the image. (We will discuss further the notion of removing "features of increasing sizes" in Sec. 2.5.) An example is provided in Fig. 14.9 and 14.10.

Schonfeld and Goutsias, 1991 noticed that besides Eqs. (14.27,14.28), *any* composition of openings γ_i and closings φ_i, *in any order*, is a filter. Their argument was generalized by Heijmans, 1997 as follows:

PROPOSITION 14.26 *Let $\psi_1, \dots, \psi_n$ be overfilters and $\xi_1, \dots, \xi_n$ be underfilters such that*

$$\psi_n \leq \cdots \leq \psi_1 \leq \xi_1 \leq \cdots \xi_n\ .$$

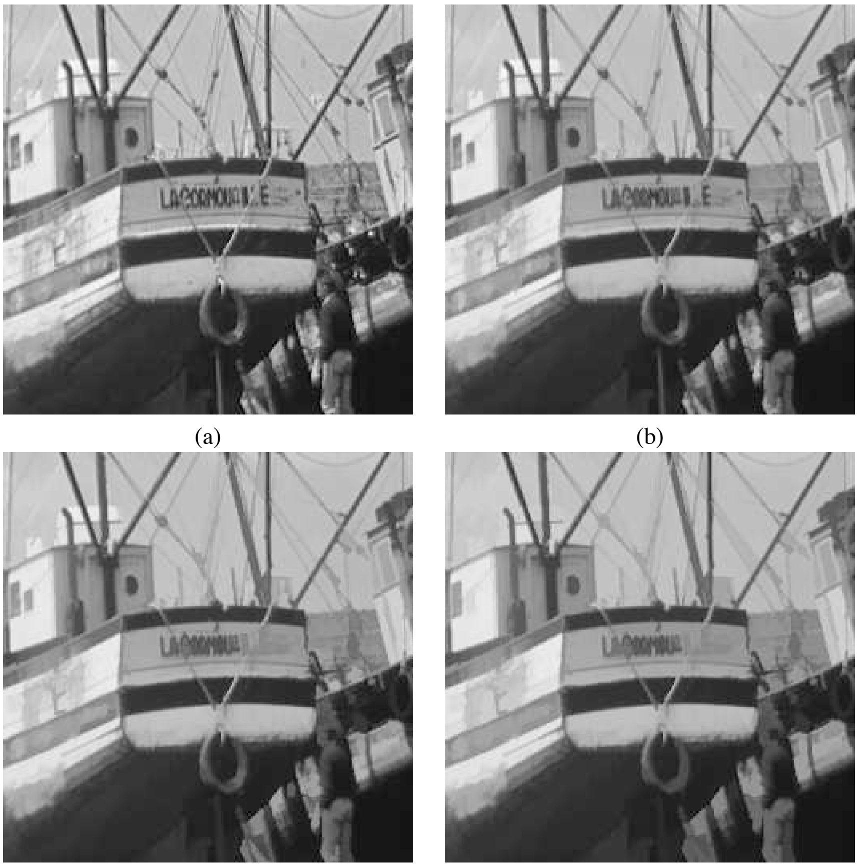
Then any composition of these operators, containing at least one ψ_i and one ξ_j, is a filter.

A consequence is the following surprising result:

PROPOSITION 14.27 *Let (ε, δ) be an adjunction in a poset L. Then any repeated composition of ε and δ in any order, containing the same number of instances of ε and of δ, is a filter.*

More precisely, an operator of the form $\psi_1 \cdots \psi_{2n}$, where for each $i = 1, \dots, 2n$ we have $\psi_i \in \{\delta, \varepsilon\}$, and $card\{i = 1, \dots, 2n \mid \psi_i = \delta\} = card\{i = 1, \dots, 2n \mid \psi_i = \varepsilon\}$, is a filter.

For more results on filters, the reader is referred to Heijmans, 1994, Heijmans, 1997, Ronse and Heijmans, 1991, Serra, 1988 and Soille, 2003.



(a) (b)

(c) (d)

Figure 14.9. Original image (a) and three steps (b, c, d) of an alternate sequential filter based on opening-closing by reconstruction using an hexagon as structuring element.

2.5 Granulometries and size distributions

As openings remove parts of an object (they are anti-extensive), one can compare two openings γ_1 and γ_2 in such terms; thus we say that γ_2 is *more active* than γ_1 if γ_2 removes from any object more than γ_1 does, in other words if $\gamma_2 \leq \gamma_1$. On the other hand, as closings add parts to an object, given two closings φ_1 and φ_2, we say that φ_2 is *more active* than φ_1 if φ_2 adds to an object more than φ_1 does, in other words if $\varphi_2 \geq \varphi_1$.

In the case of the complete lattice $\mathcal{P}(E)$, given two structuring elements $A, B \in \mathcal{P}(E)$, we define the relation $\sqsupseteq$ by $B \sqsupseteq A$ iff B is a union of translates

(a) (b)

Figure 14.10. Steps 1 (a) and 3 (b) of the same filter as in Fig. 14.9 but using segments in different directions as structuring elements.

of A. Readily, by Eqs. (14.1,14.4) we have

$$B \sqsupseteq A \iff B \circ A = B \iff \big(\exists C \in \mathcal{P}(E),\ B = A \oplus C\big) \ .$$

For example, given $b \geq a$, this is true if A and B are squares of size a and b respectively, or (for $E = \mathbb{R}^n$) if A and B are closed balls of radii a and b respectively. Now by Equation (14.4), the openings γ_A, γ_B and closings φ_A, φ_B verify:

$$\gamma_B \leq \gamma_A \iff \varphi_B \geq \varphi_A \iff B \sqsupseteq A \ .$$

In other words, the "greater" is the structuring element (for $\sqsupseteq$), the more active are the opening and closing.

The above suggests that the activity of openings and closings is governed by the size of the structuring elements that they use. We see below that it can be characterized in another way:

PROPOSITION 14.28 *Let ψ_1 and ψ_2 be either two openings, or two closings, on a poset L. Then the following four statements are equivalent:*

1 ψ_2 *is more active than* ψ_1.

2 $Inv(\psi_2) \subseteq Inv(\psi_1)$.

3 $\psi_2\psi_1 = \psi_2$.

4 $\psi_1\psi_2 = \psi_2$.

The second item indicates that the activity increases as the domain of invariance decreases. The last two suggest a notion of a filtering absorption order associated to activity: if ψ_2 is more active than ψ_1, then as a filter ψ_2 is more severe, so ψ_1 does not improve in any way upon the result of ψ_2, whether applied before of after it; thus ψ_2 *absorbs* ψ_1. We can now consider an ordered sequence of openings:

DEFINITION 14.29 *A* granulometry *(on a poset L) is a family of operators ψ_r ($r \in R \subseteq \mathbb{R}^+$) such that:*

1 $\forall r \in R$, ψ_r is anti-extensive;

2 $\forall r \in R$, ψ_r is increasing;

3 $\forall r, s \in R$, $\psi_r \psi_s = \psi_s \psi_r = \psi_{\max(r,s)}$.

Applying item 3 with $r = s$, ψ_r is idempotent, so it is an opening. In fact, ψ_r ($r \in R$) is a granulometry iff ψ_r is an opening for every r, and ψ_r decreases (becomes more active) as the parameter r increases: $\forall r, s \in R$, $r \geq s$ implies $\psi_r \leq \psi_s$, or equivalently $Inv(\psi_r) \subseteq Inv(\psi_s)$.

For binary images in a digital framework ($L = \mathcal{P}(\mathbb{Z}^n)$), we take $R = \{2, \ldots, r_{max}\}$ and ψ_r to be the opening by a structuring element B_r corresponding to size r (say, a $r \times r$-square). Then for a set $X \subseteq \mathbb{Z}^n$, it is interesting to measure the area (number of pixels) of $\gamma_r(X)$ for all r; this gives a decreasing function $R \to \mathbb{N}$, the *granulometry curve* of X, it displays the area of the objects according to the size of the opening. Positions where this curve decreases sharply indicate that there are substantial parts of X having the corresponding width. This is illustrated in Fig. 14.11.

One defines similarly an *anti-granulometry* by replacing, in item 1 of Definition 14.29, "anti-extensive" by "extensive". Then ψ_r ($r \in R$) is an anti-granulometry iff ψ_r is a closing for every r, and ψ_r increases (becomes more active) as the parameter r increases: $\forall r, s \in R$, $r \geq s$ implies $\psi_r \geq \psi_s$, or equivalently $Inv(\psi_r) \subseteq Inv(\psi_s)$. In a digital framework, we define the *anti-granulometry curve*, which gives an indication on the width of the holes of the set.

It is possible to combine a granulometry γ_r ($r \in R_1 \subseteq \mathbb{R}^+$) and an anti-granulometry φ_r ($r \in R_2 \subseteq \mathbb{R}^+$) into a two-sided sequence ψ_r, $r \in R = R_1 \cup \{0\} \cup (-R_2)$ by setting $\psi_r = \gamma_r$ for $r \in R_1$, $\psi_0 = \mathbf{id}$, and $\psi_{-r} = \varphi_r$ for $r \in R_2$. Then the axioms are: (1) ψ_r is anti-extensive for $r \geq 0$ but extensive for $r \leq 0$, (2) ψ_r is increasing, (3) $\psi_r \psi_s = \psi_s \psi_r = \psi_{m(r,s)}$ for r, s having the same sign, where $m(r, s) = \max(r, s)$ for $r, s \geq 0$, but $m(r, s) = \min(r, s)$ for $r, s \neq 0$ (we have no such identity for $r > 0$ and $s < 0$). We generalize then the granulometry curve into a function $R \to \mathbb{N}$ where the parts $r < 0$ and $r > 0$ deal with the sizes of holes and grains respectively.

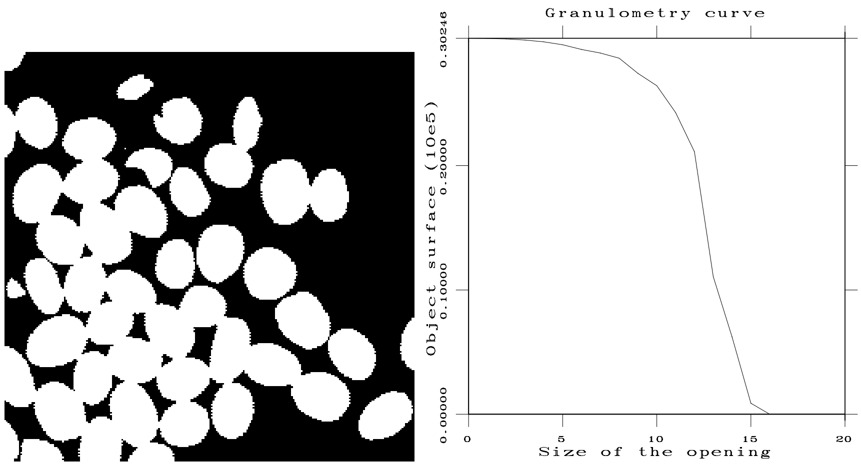


Figure 14.11. A binary image of coffee beans, and its granulometry curve, showing a sharp decrease between 10 and 15; this indicates that most beans have such a width.

3. Related approaches

As we said in the Introduction, there are other fields of research which are based on the same lattice-theoretical foundations as mathematical morphology. We present here three of them, which we think are relevant to the logic of spatial relations: formal concept analysis, rough sets and fuzzy sets. Roughness can be represented by using mathematical morphology operators to define upper and lower approximations in the framework of roughs sets (Sec. 3.2). We then show why mathematical morphology can be considered as a spatial reasoning tool (Sec. 3.3), with its two components: spatial knowledge representation and reasoning. As for the first one, we present in Sec. 3.4 an extension of mathematical morphology to fuzzy sets, which leads to an extended representation power coping with spatial imprecision. Modeling spatial relationships based on fuzzy morphology allows reasoning under imprecision and with structural spatial information, as shown in Sec. 3.5. This reasoning component will be further explored in Sec. 4.

3.1 Formal Concept Analysis

It is a lattice-based theory (Ganter and Wille, 1999) of relations between objects and features. It can be applied to spatial objects and spatial relations like visibility, enclosure, etc.

Let Ω be a set of *objects*, Π a set of *properties*, and $\sim$ a relation between Ω and Π, where $o \sim p$ means that object o has property p. The triple $(\Omega, \Pi, \sim)$ is called a *context*. Following Theorem 14.12 we define the two maps

$$\begin{array}{rll} \omega: & \mathcal{P}(\Pi) \to \mathcal{P}(\Omega): & P \mapsto \{o \in \Omega \mid \forall p \in P,\ o \sim p\} \ , \\ \pi: & \mathcal{P}(\Omega) \to \mathcal{P}(\Pi): & O \mapsto \{p \in \Pi \mid \forall o \in O,\ o \sim p\} \ . \end{array}$$

Thus $\omega(P)$ is the set of objects sharing all properties in P, while $\pi(O)$ is the set of properties shared by all objects in O. A *concept* is a pair (O, P), where $O \in \mathcal{P}(\Omega)$ and $P \in \mathcal{P}(\Pi)$, such that $O = \omega(P)$ and $P = \pi(O)$; O is the *extent* of the concept and P is the *intent* of the concept.

The set of concepts ordered by inclusion of object sets, or equivalently by the inverse inclusion on property sets:

$$(O_1, P_1) \leq (O_2, P_2) \iff O_1 \subseteq O_2 \iff P_1 \supseteq P_2 \ .$$

It forms then a complete lattice with the following supremum and infimum operations:

$$\begin{array}{rcl} \bigvee_{i \in I} (O_i, P_i) & = & \left(\omega\pi\left[\bigcup_{i \in I} O_i\right], \bigcap_{i \in I} P_i\right) \ , \\ \bigwedge_{i \in I} (O_i, P_i) & = & \left(\bigcap_{i \in I} O_i, \pi\omega\left[\bigcup_{i \in I} P_i\right]\right) \ . \end{array}$$

Here we used the complete lattice structure of the two Moore families of extents (possible O_i's) and of intents (possible P_i's), cf. Proposition 14.9 and Theorem 14.5.

A possible example of application of formal concept analysis is to consider two subsets S and V of the space $\mathbb{R}^n$ or $\mathbb{Z}^n$, where S represents an object to be visually examined, and a V is a set of viewpoints. We define a relation $\sim$ between the boundary ∂S of S and V, namely $s \sim v$ if s is visible from v, i.e., the open segment $]s, v[$ is disjoint from S. Then a concept is given by a pair (T, W), where $T \subseteq \partial S$ and $W \subseteq V$, such that T is the set of positions visible by all viewpoints in W, and W is the set of all viewpoints from which T is entirely visible.

3.2 Mathematical morphology and rough sets

Rough set theory has been introduced by Pawlak, 1982, as an extension of set theory, mainly in the domain of intelligent systems. The objective was to deal with incomplete information, leading to the idea of indistinguishability of objects in a set. It is therefore related to the concept of approximation, and of granularity of information (in the sense of Zadeh, 1979). This theory was applied successfully in several applications, e.g. information analysis, data analysis and data mining, knowledge discovery (for instance discovery of which features are relevant for data description), i.e., all applications for which a need arises for intelligent decision support. Let us mention in particular the works of Lin, 1995, Lin and Liu, 1994, Yao, 1998 and Yao and Lin, 1996. There have also been studies towards a fuzzy approach to rough sets (Dubois and Prade, 1990), and on their relations with logic (Orlowska, 1993; Pawlak, 1987).

In this framework, a set X is approximated by two sets, called *upper* and *lower approximations*, and denoted by $\overline{A}(X)$ and $\underline{A}(X)$, such that $\underline{A}(X) \subseteq X \subseteq \overline{A}(X)$. It is interesting to investigate the algebraic properties of the two set operators $\overline{A}$ and $\underline{A}$: the first one is extensive, the second one is anti-extensive, and probably they should be increasing. But then, are they respectively a dilation and an erosion? In particular, do they constitute an adjunction (arising from a reflexive relation, cf. Proposition 14.21)? Or are they a closing and an opening? Or is $\overline{A}$ both a closing and a dilation, and $\underline{A}$ both an opening and an erosion (cf. Proposition 14.22)? Are they dual by complementation? Surprisingly, there have not been many studies on the relation between rough sets and MM; let us mention a few of them. On the one hand Polkowski, 1998 built a hit-or-miss (Fell) topology on rough sets, similar to the one used in MM for closed sets (Heijmans, 1994; Matheron, 1975; Ronse and Tajine, 2004; Serra, 1982). On the other hand Bloch, 2000b studied the algebraic properties of the upper and lower approximation operators, and established an analogy between them and the classical morphological operators on Euclidean or digital sets, namely dilation, erosion, opening and closing by a structuring element. Also Düntsch and Gediga, 2003 considered the algebraic aspects of rough sets, in particular their links with Galois connections.

Here we will investigate rough sets in light of the theory of adjunctions on sets; in some sense, this is a generalization of Bloch, 2000b. At the same time we will address topological aspects. But let us first recall the basic definitions of rough sets, in particular those based on a similarity relation.

In rough set theory (Pawlak, 1982), the two approximations $\overline{A}(X)$ and $\underline{A}(X)$ such that $\underline{A}(X) \subseteq X \subseteq \overline{A}(X)$ are defined from an equivalence relation. Let $\mathcal{U}$ denote the universe of discourse (X being a subset of $\mathcal{U}$). We consider *attributes* which are functions defined on $\mathcal{U}$, and write A for the set of attributes. To each $x \in \mathcal{U}$ we associate an *information vector* $Inf(x)$, which is the set of attributes associated to x. We define an equivalence relation R_A (with respect to the set A of attributes on $\mathcal{U}$) by the equality of the information vector:

$$x \; R_A \; y \iff Inf(x) = Inf(y) \ .$$

Assuming that each element of $\mathcal{U}$ is known only through its attributes, $x \; R_A \; y$ means that x and y are undistinguishable on the basis of available information. The pair $(\mathcal{U}, R_A)$ is called an approximation space. For $x \in \mathcal{U}$, let $[x]_A$ denote the equivalence class of x under R_A:

$$[x]_A = \{y \in \mathcal{U} \mid x \; R_A \; y\} \ .$$

Then upper and lower approximations of a subset X of $\mathcal{U}$ are defined as:

(14.29)
$$\begin{array}{rcl}\overline{A}(X) &=& \{x \in \mathcal{U} \mid [x]_A \cap X \neq \emptyset\} \ , \\ \underline{A}(X) &=& \{x \in \mathcal{U} \mid [x]_A \subseteq X\} \ .\end{array}$$

The lower approximation of X contains all points of $\mathcal{U}$ that are distinguishable from every elements of X^c, while its upper approximation contains all points of $\mathcal{U}$ that are undistinguishable from some element of X. We call a *rough set* a pair $(\underline{A}(X), \overline{A}(X))$.

Let us refer to the terminology used for dilations and erosions on sets: if R_A stands for ρ, then by Eqs. (14.24,14.25,14.26) we have $[x]_A = N(x)$, $\underline{A} = \varepsilon_N = \varepsilon_{R_A}$ and $\overline{A} = \delta_{\widetilde{N}} = \delta_{R_A^{-1}}$. Clearly $\overline{A}$ and $\underline{A}$ are dual under complementation: $\overline{A}(X) = \left[\underline{A}(X^c)\right]^c$. The fact that R_A is symmetrical ($R_A = R_A^{-1}$) means that $N = \widetilde{N}$, so $(\underline{A}, \overline{A})$ forms an adjunction. Now as R_A is reflexive and transitive, by Proposition 14.22, the erosion $\underline{A}$ is also an opening, while the dilation $\overline{A}$ is also a closing, with $\overline{A}\underline{A} = \underline{A}$ and $\underline{A}\overline{A} = \overline{A}$. In particular, we have

$$\forall X \in \mathcal{P}(\mathcal{U}), \qquad \underline{A}(X) \subseteq X \subseteq \overline{A}(X) \ .$$

By Proposition 14.22 again, $\varepsilon_N(\mathcal{P}(\mathcal{U})) = \delta_N(\mathcal{P}(\mathcal{U})$, i.e., the families of lower and upper approximations coincide:

$$\{\underline{A}(X) \mid X \in \mathcal{P}(\mathcal{U})\} = \{\overline{A}(X) \mid X \in \mathcal{P}(\mathcal{U})\} \ ;$$

in fact this family consists of all sets which are unions of equivalence classes $[x]_A$. With the topological interpretation given after Proposition 14.22, it constitutes an Alexandroff topology on $\mathcal{U}$, where open sets coincide with closed ones, the upper approximation $\overline{A}(X)$ is the closure and the star (least open superset) of X, while the lower approximation $\underline{A}(X)$ is the interior and the greatest closed subset of X.

This definition can be extended to any relation R, leading to the notion of generalized approximate space (see e.g. Yao, 1998). Simply we take an arbitrary relation R instead of the equivalence R_A, and the set

$$r(x) = \{y \in \mathcal{U} \mid x \ R \ y\}$$

instead of the equivalence class $[x]_A$; here $r(x)$ corresponds to the set $N(x)$ according to Equation (14.25), with R standing for ρ. Then Equation (14.29) becomes

(14.30)
$$\begin{array}{rcl}\overline{R}(X) &=& \{x \in \mathcal{U} \mid r(x) \cap X \neq \emptyset\} \ , \\ \underline{R}(X) &=& \{x \in \mathcal{U} \mid r(x) \subseteq X\} \ .\end{array}$$

In our terminology, $\underline{R} = \varepsilon_r = \varepsilon_R$ and $\overline{R} = \delta_{\widetilde{r}} = \delta_{R^{-1}}$. Now $\overline{R}$ and $\underline{R}$ are still dual under complementation.

Equivalently, we can define $\underline{R}$ and $\overline{R}$ as two set operators which are dual under complementation ($\overline{R}(X) = \left[\underline{R}(X^c)\right]^c$), and such that $\underline{R}$ is an erosion (or equivalently: $\overline{R}$ is a dilation). This is in accordance with the operator-oriented view of rough sets (Lin and Liu, 1994; Yao, 1998).

If R is an equivalence relation, we get Pawlak's definition, Equation (14.29); indeed, we can define $Inf(x) = [x]_R$, the equivalence class of x under R. Let us consider weaker conditions on R. We require that $\underline{R}(X) \subseteq \overline{R}(X)$ for all $X \in \mathcal{P}(\mathcal{U})$, which means that $\varepsilon_r \leq \delta_{\widetilde{r}}$; according to Proposition 14.20, this is verified iff

$$\forall x \in \mathcal{U}, \qquad r(x) \neq \emptyset \ .$$

Usually, one requires that $\underline{R}$ is anti-extensive and $\overline{R}$ is extensive, that is,

$$\forall X \in \mathcal{P}(\mathcal{U}), \qquad \underline{R}(X) \subseteq X \subseteq \overline{R}(X) \ ;$$

by Proposition 14.21, this is verified iff R is reflexive.

By Proposition 14.21, R is symmetrical iff $\underline{R}\overline{R}$ is extensive, iff $\overline{R}\underline{R}$ is anti-extensive. Then $r = \widetilde{r}$, and $(\underline{R}, \overline{R})$ is an adjunction. If R is reflexive and symmetrical, we call it a *tolerance relation*.

By Proposition 14.22, R is reflexive and transitive iff $\underline{R}$ is both an erosion and an opening, iff $\overline{R}$ is both a dilation and a closing. Here we have an Alexandroff topology on $\mathcal{U}$, where for every $X \in \mathcal{P}(\mathcal{U})$, $\underline{R}(X)$ is the topological interior of X and $\overline{R}(X)$ is the topological closure of X. The family of open sets, $\{\underline{R}(X) \mid X \in \mathcal{P}(\mathcal{U})\}$, and the family of closed sets,$\{\overline{R}(X) \mid X \in \mathcal{P}(\mathcal{U})\}$, do not coincide, unless R is also symmetrical (i.e., an equivalence relation).

When R is reflexive but not transitive, the upper and lower approximations do generally not correspond to a topology. However they correspond to the closure and interior in a *pre-topology*, that is: $\overline{R}$ and $\underline{R}$ are dual under complementation, they are both increasing, $\overline{R}$ is extensive while $\underline{R}$ is anti-extensive, $\overline{R}(\emptyset) = \emptyset$ and $\underline{R}(\mathcal{U}) = \mathcal{U}$. Let us mention the use by Emptoz, 1983 of pre-topology for the description of spatial objects. This may be of interest for pattern recognition purposes, since a non-idempotent closure allows to aggregate patterns using iterated closure operations.

In Yao, 1998, various properties are given for the operators $\underline{R}$ and $\overline{R}$, which may or may not be satisfied, according to the properties of the relation R. In Bloch, 2000b, a parallel is made between these properties and those of dilations, erosions, openings and closings. In fact, *all* these properties follow from the ones given in Sec. 2.3 for dilations and erosions on sets, those of openings and closings in Sec. 2.2, and from Equation (14.15).

Using the operator-oriented point of view (Lin and Liu, 1994; Yao, 1998), one could also define the lower and upper approximation as an opening and a closing. However, such operators cannot be defined in terms of a relation or a neighbourhood function, as was the case with dilations and erosions. Openings and closings are characterized by their invariance domain: for an opening, it is a dual Moore family, i.e., a family closed under arbitrary unions and containing $\emptyset$; for a closing it is a More family, i.e., a family closed under arbitrary intersections and containing $\mathcal{U}$. Let L be an opening and let $\mathcal{S}$ be a family of nonvoid parts of $\mathcal{U}$ such that $Inv(L)$ is the set of all arbitrary unions of elements of $\mathcal{S}$ (including the empty union $\emptyset$); in fact L is the opening $\gamma_{\mathcal{S}}$ defined after Eqs. (14.18,14.19), as for every $X \in \mathcal{P}(\mathcal{U})$ we have

$$L(X) = \bigvee\{A \in \mathcal{S} \mid A \subseteq X\} \ .$$

For example, if $\mathcal{U} = E$ and L is the opening by a structuring element B, then $\mathcal{S}$ is the set of translates B_p ($p \in E$) of B. Now let H be the closing which is the dual of L by complementation; then $Inv(H) = \{A^c \mid A \in Inv(L)\}$, and $Inv(H)$ is the set of arbitrary intersections of A^c for $A \in \mathcal{S}$. For $X \in \mathcal{P}(\mathcal{U})$ we have

$$H(X) = \left(\bigvee\{A \in \mathcal{S} \mid A \subseteq X^c\}\right)^c \ .$$

We can interpret the elements of $\mathcal{S}$ as "blocks", and a point x can be included in the lower approximation $L(X)$ only through its membership of a "block" included in X, while a point x can be excluded from the upper approximation $H(X)$ only through its membership of a "block" excluded from X. However these blocks do not make a partition, as with Pawlak's definition, Equation (14.29).

An interesting particular case is when L and H are the interior and closure operators in a topological space. One speaks then of a *topological approximation space*. As explained after Proposition 14.22, this is weaker than requiring H to be both a dilation and a closing, and dually L to be both an erosion and an opening.

Other operators could be used for lower and upper approximations. In Serra, 1988, it is shown that every increasing operator on a complete lattice, which fixes the greatest element, is a supremum of erosions. Thus an increasing operator ψ on $\mathcal{P}(\mathcal{U})$ such that $\psi(\mathcal{U}) = \mathcal{U}$, is a union of erosions; in particular ψ is anti-extensive iff these erosions are anti-extensive. Consider the following definition of lower and upper approximations (Lin, 1995). Suppose that to each $x \in \mathcal{U}$ we associate a family $\mathcal{N}(x)$ of parts of $\mathcal{U}$ which are "neighbourhoods" of x; $\mathcal{N}(x)$ is called a *neighbourhood system of* x. Now we define the upper and lower approximations $\overline{N}$ and $\underline{N}$ as follows:

$$\begin{aligned} \overline{N}(X) &= \{x \in \mathcal{U} \mid \forall A \in \mathcal{N}(x), A \cap X \neq \emptyset\} \ , \\ \underline{N}(X) &= \{x \in \mathcal{U} \mid \exists A \in \mathcal{N}(x), A \subseteq X\} \ . \end{aligned} \tag{14.31}$$

Clearly $\overline{N}$ is the dual by complementation of $\underline{N}$, so let us analyse the latter. We have

$$\underline{N}(\mathcal{U}) = \{x \in \mathcal{U} \mid \mathcal{N}(x) \neq \emptyset\} \ ,$$

so in order to have $\underline{N}(\mathcal{U}) = \mathcal{U}$, we suppose that $\mathcal{N}(x) \neq \emptyset$ for all $x \in \mathcal{U}$. Then $\underline{N}$ is a union of erosions, and we can describe them precisely. For every $x \in \mathcal{U}$ and for every $A \in \mathcal{N}(x)$, let $N[x, A] : \mathcal{U} \to \mathcal{P}(\mathcal{U})$ be defined by $N[x, A](x) = A$ and $N[x, A](y) = \mathcal{U}$ for $y \in \mathcal{U} \setminus \{x\}$. Then $\varepsilon_{N[x,A]}$ verifies for every $X \in \mathcal{P}(\mathcal{U})$:

$$\varepsilon_{N[x,A]}(X) = \begin{cases} \mathcal{U} & \text{if } X = \mathcal{U} \ , \\ \{x\} & \text{if } A \subseteq X \neq \mathcal{U} \ , \\ \emptyset & \text{otherwise} \ . \end{cases}$$

We obtain then

$$\underline{N}(X) = \bigcup_{x \in \mathcal{U}} \bigcup_{A \in \mathcal{N}(X)} \varepsilon_{N[x,A]}(X) \ .$$

Dually, we get

$$\overline{N}(X) = \bigcap_{x \in \mathcal{U}} \bigcap_{A \in \mathcal{N}(X)} \delta_{\widetilde{N[x,A]}}(X) \ .$$

This view is particularly interesting for shape recognition, since in morphological recognition, an object has often to be tested or matched with a set of patterns, like directional structuring elements. Thus we apply the union of the erosions by all those patterns, which is a particular case of the above operator.

In Bloch, 2000b, some other extensions are presented, using as lower and upper approximations morphological thinning and thickening (Serra, 1982). There is also an extension to rough functions, using the grey-level morphological operations described in Sec. 1.2.

To conclude, let us remark that the general theory of adjunctions, dilations, erosions, openings and closings on sets provides a good formal framework for expressing the notion of coarseness underlying rough sets. It allows to characterize precisely their algebraic properties and their relations with topology. In Sec. 3.4 and 4, we will examine the relationship of rough sets with fuzzy sets and modal logic, especially in the morphological framework.

3.3 Mathematical morphology and spatial reasoning

Spatial reasoning has been largely developed in artificial intelligence, in particular using qualitative representations based on logical formalisms. In image interpretation and computer vision it is much less developed and is mainly based on quantitative representations. On the contrary, mathematical morphology is widely used in these domains. A typical example concerns model-based structure recognition in images, where the model represents spatial entitites and relationships between them. Based on this example, spatial reasoning can be defined as the domain of spatial knowledge representation, in particular spatial relations between spatial entities, and of reasoning on these entities and relations. This definition exhibits two main components: spatial knowledge representation and reasoning. In particular spatial relationships constitute an important part of the knowledge we have to handle and imprecision is often attached to it. The reasoning component includes fusion of heterogeneous spatial knowledge, decision making, inference, recognition. Two types of questions are raised when dealing with spatial relationships:

1 given two objects (possibly fuzzy), assess the degree to which a relation is satisfied;

2 given one reference object, define the area of space in which a relation to this reference is satisfied (to some degree).

In order to answer these questions and address both representation and reasoning issues, different frameworks and their combination can be used. Fuzzy set theory has powerful features to represent imprecision at different levels, to combine heterogeneous information and to make decision (Dubois and Prade, 1985; Dubois et al., 1999). Formal logics and the attached reasoning and inference power are widely used too, usually in a qualitative context. But mathematical morphology, which is an algebraic theory that has extensions to fuzzy sets and to logical formulas, is a very promising tool, since it can elegantly unify the representation of several types of relationships (Bloch, 2003b). The association of different frameworks for spatial reasoning allows us to match two requirements such as axiomatization, expressiveness and completeness with respect to the types of spatial information we want to represent (Aiello, 2002). Complexity issues are not addressed here, but it should be noted that efficient algorithms exist for digital morphology.

Mathematical morphology provides tools for spatial reasoning at several levels. It provides tools for representing objects or object features (see e.g. Sec. 1.1 and 2). For instance skeletons provide compact and expressive representations of shapes; morphological tools for shape decomposition lead to structured representations, such as graphs for instance; spatial imprecision can be represented by a pair of dilation and erosion; tools for selecting objects or parts of objects having specific properties can be derived from morphological operators such as hit-or-miss transformations for instance, etc. These aspects, quite traditional in mathematical morphology, are not detailed here, and we will concentrate rather on tools for representing spatial relations. The notion of structuring element captures the local spatial context and leads to analysis of a scene using operators involving the neighbourhood of each point. At a more global level, several spatial relations between spatial entities can be expressed as morphological operations, in particular using dilations. Let us provide a few examples, of metric and topological relationships.

Several distances between objects can be expressed in terms of dilation. The minimum or nearest point distance between two sets X and Y is defined (in the discrete finite case) as:

$$\begin{aligned} d_N(X,Y) &= \min_{(x,y)\in X\times Y} d_E(x,y) \\ &= \min_{x\in X} d_E(x,Y) = \min_{y\in Y} d_E(y,X), \end{aligned} \tag{14.32}$$

where d_E denotes the Euclidean distance in $\mathcal{S}$ (note that this function is improperly named distance since it is not separable and does not satisfy the triangular inequality). This has an equivalent morphological expression:

$$\begin{aligned} d_N(X,Y) &= \inf\{n \in \mathbb{N}, X \cap \delta_n(Y) \neq \emptyset\} \\ &= \inf\{n \in \mathbb{N}, Y \cap \delta_n(X) \neq \emptyset\}. \end{aligned} \tag{14.33}$$

Another morphological expression is, for $n > 0$:

$$d_N(X,Y) = n \Leftrightarrow \delta_n(X) \cap Y \neq \emptyset \text{ and } \delta_{(n-1)}(X) \cap Y = \emptyset \tag{14.34}$$

or equivalently the symmetrical expression. For $n = 0$ we have:

$$d_N(X,Y) = 0 \Leftrightarrow X \cap Y \neq \emptyset. \tag{14.35}$$

The Hausdorff distance is defined as:

$$H_d(X,Y) = \max[\sup_{x\in X} d_E(x,Y), \sup_{y\in Y} d_E(y,X)]. \tag{14.36}$$

Similarly as for the nearest point distance, this distance can be expressed in morphological terms as:

$$H_d(X,Y) = \inf\{n, X \subseteq \delta_n(Y) \text{ and } Y \subseteq \delta_n(X)\}. \tag{14.37}$$

Alternatively, we can write:

$$H_d(X,Y) = 0 \Leftrightarrow X = Y, \tag{14.38}$$

and for $n > 0$:

$$\begin{aligned} &H_d(X,Y) = n \Leftrightarrow X \subseteq \delta_n(Y) \text{ and } Y \subseteq \delta_n(X) \\ &\quad \text{and } \left(X \not\subseteq \delta_{(n-1)}(Y) \text{ or } Y \not\subseteq \delta_{(n-1)}(X)\right). \end{aligned} \tag{14.39}$$

From these representations, several types of knowledge about distance can be expressed. For instance, Fig. 14.12 shows a spatial representation of "B is at a distance between n_1 and n_2 from A", i.e. B should be in the dilation of radius n_2 of A but not in the dilation of radius n_1 of A.

Another example is adjacency. Here, we restrict ourselves to the digital case, and use discrete topology as derived from digital connectivity for defining

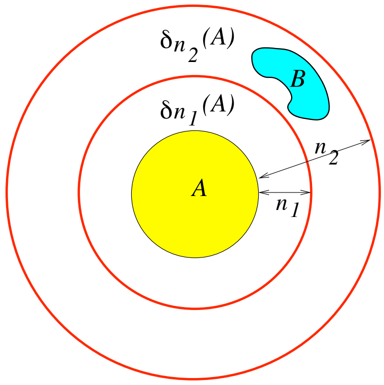


Figure 14.12. Illustration of a distance relation expressed by an interval.

adjacency between two regions X and Y, subsets of the digital space (see Ch. 12 of this book for further details about digital topology). Let us consider an n-dimensional digital space (typically $\mathbb{Z}^n$), and any discrete connectivity defined on this space, denoted c-connectivity (for instance, for $n = 3$, we may consider 6-, 18- or 26-connectivity on a cubic grid). Let $n_c(x, y)$ be the Boolean variable stating that x and y are neighbours in the sense of the discrete c-connectivity. Let B_c be the set of c-neighbours of the origin. For any two subsets X and Y in $\mathbb{Z}^n$, X and Y are adjacent according to the c-connectivity if: $X \cap Y = \emptyset \ \ and \ \ \exists x \in X, \exists y \in Y : n_c(x, y)$.

This definition can also be expressed equivalently in terms of morphological dilation, as:

$$X \cap Y = \emptyset \ \text{ and } \ \delta_{B_c}(X) \cap Y \neq \emptyset, \tag{14.40}$$

where $\delta_{B_c}(X)$ denotes the dilation of X by the structuring element B_c.

Another topological relation, often used in the context of mereotopology for instance, is tangential proper part. Again this can be expressed in morphological terms, as illustrated in Fig. 14.13.

These expressions extend to different frameworks, including fuzzy set theory and formal logics, thus benefiting from the reasoning power of these frameworks.

3.4 Fuzzy mathematical morphology

Dealing with spatial imprecision can be adequately addressed by defining objects or regions of space as fuzzy objects. We denote by $\mathcal{S}$ the spatial domain, typically $\mathbb{Z}^2$ or $\mathbb{Z}^3$ for digital 2D or 3D images, or, in the continuous case, $\mathbb{R}^2$ or $\mathbb{R}^3$. A fuzzy object is a fuzzy set defined on $\mathcal{S}$, i.e. a spatial fuzzy set. Its

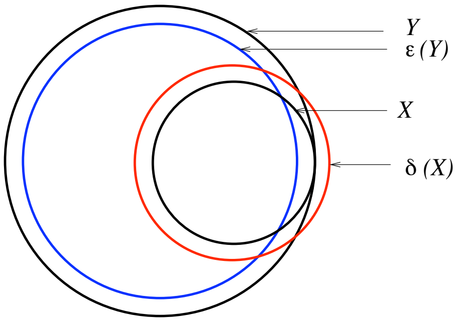


Figure 14.13. Illustration of tangential part relationship, and its expression in terms of dilation and erosion: X is included in Y while its dilation is not (equivalently, X is not included in the erosion of Y).

membership function μ represents the imprecision in the spatial extent of the object. For any point x of $\mathcal{S}$, $\mu(x)$ is the degree to which x belongs to the fuzzy object. We denote by $\mathcal{F}$ the set of fuzzy sets on $\mathcal{S}$.

Several definitions of fuzzy mathematical morphology have been proposed since a few years. Some of them just consider grey level as membership functions (Goetcherian, 1980; Giardina and Sinha, 1989; Laplante and Giardina, 1991; di Gesu, 1988; di Gesu et al., 1993; Nakatsuyama, 1993), or use binary structuring elements (Rosenfeld, 1984). Here we restrict the presentation to really fuzzy approaches, where fuzzy sets have to be transformed according to fuzzy structuring elements. Initial developments can be found in the definition of fuzzy Minkowski addition (Dubois and Prade, 1983; Kaufmann and Gupta, 1988). Then this problem has been addressed by several authors independently (Bloch, 1993; Bloch and Maître, 1995; Sinha and Dougherty, 1992; de Baets and Kerre, 1993; de Baets, 1995; Bandemer and Näther, 1992; Popov, 1995; Sinha et al., 1997; Nachtegael and Kerre, 2000; Maragos et al., 2001; Deng and Heijmans, 2002).

Attention will be paid here only to the 4 basic operations of mathematical morphology (erosion, dilation, opening, closing), but it should be clear for the reader that for every definition, a complete set of morphological operations could be derived. Extensions of mathematical morphology have been proposed for instance for defining more complex operations (like filtering) (Bloch and Maître, 1995; Sinha et al., 1997), and geodesic operations (Bloch, 2000a).

Although a fuzzy set is defined through its membership function, functional approaches are not appropriate. For instance, the classical dilation of a function taking values in $[0, 1]$ by a functional structuring element taking values in $[0, 1]$ generally provides a function with values in $[0, 2]$ which has no direct interpretation in terms of fuzzy sets. Therefore a set theoretical approach is preferred, where set operations are converted into their fuzzy equivalents, thus preserving the compatibility with classical morphology in case the fuzzy sets reduce to crisp ones.

General methods for extending an operation to fuzzy sets. Common and generic methods that can be used for defining a fuzzy operator or fuzzy relationship from the corresponding binary ones can be categorized in three main classes. The first type relies on the “extension principle”, as introduced by Zadeh (Zadeh, 1975; Dubois and Prade, 1980). The second class relies on computation on α-cuts (e.g. Dubois and Jaulent, 1987; Krishnapuram et al., 1993; Bloch and Maître, 1995). These two classes of definitions explicitly involve the operations or relations on crisp sets. The third class of methods consists in providing directly fuzzy definitions of the operations or of the relationships, by substituting all crisp expressions by their fuzzy equivalents. This type of translation is used in the following.

This translation is generally done term by term. For instance, intersection is replaced by a t-norm, union by a t-conorm, sets by fuzzy set membership functions, etc. A triangular norm (or t-norm) is a function from $[0,1] \times [0,1]$ into $[0,1]$ which is commutative, associative, increasing, and for which 1 is unit element and 0 is null element (Menger, 1942; Schweizer and Sklar, 1963). Examples of t-norms are min, product, etc. (Dubois and Prade, 1980). A t-conorm is its dual with respect to complementation. This type of translation is particularly straightforward if the binary relationship can be expressed in set theoretical and logical terms.

Let us take a simple example to illustrate this method. A fuzzy set μ is said to be included in another fuzzy set ν if $\forall x \in \mathcal{S},\ \mu(x) \leq \nu(x)$. This is a crisp definition of inclusion of fuzzy sets. We may also consider that if two sets are imprecisely defined, their inclusion relationship may be imprecise too. Therefore inclusion of fuzzy sets becomes a matter of degree. This degree of inclusion can be obtained using the translation principle. In the crisp case, the set equation expressing inclusion of a set X in a set Y can be written as follows:

$$X \subseteq Y \Leftrightarrow X^c \cup Y = \mathcal{S} \Leftrightarrow \forall x \in \mathcal{S},\ x \in X^c \cup Y. \tag{14.41}$$

In the fuzzy case, X and Y become fuzzy sets having membership functions μ and ν and we have the following correspondences:

$$\forall x \in \mathcal{S} \quad \leftrightarrow \quad \inf_{x \in \mathcal{S}}, \tag{14.42}$$

$$x \in X^c \quad \leftrightarrow \quad c[\mu(x)], \tag{14.43}$$

$$x \in Y \quad \leftrightarrow \quad \nu(x), \tag{14.44}$$

$$X^c \cup Y \quad \leftrightarrow \quad T[c(\mu), \nu]. \tag{14.45}$$

Finally, the degree of inclusion of μ in ν is defined as:

$$\mathcal{I}(\mu,\nu) = \inf_{x\in\mathcal{S}} T[c(\mu(x)),\nu(x)], \tag{14.46}$$

where T is a t-conorm and c a fuzzy complementation.

Fuzzy morphology by formal translation based on t-norms and t-conorms. The first attempts to build fuzzy mathematical morphology based on this translation principle were developed in Bloch, 1993 and Bloch and Maître, 1995, and coincide with the definitions independently developed in Bandemer and Näther, 1992. An important property that was put to the fore in this approach is the duality between erosion and dilation.

From the following set equivalence (where $\varepsilon_B(X)$ denotes the erosion of the set X by B): $x \in \varepsilon_B(X) \Leftrightarrow B_x \subseteq X$, a natural way to define the erosion of a fuzzy set μ by a fuzzy structuring element ν is to use the degree of inclusion defined above:

$$\forall x \in \mathcal{S},\ \varepsilon_\nu(\mu)(x) = \inf_{y\in\mathcal{S}} T[c(\nu(y-x)),\mu(y)]. \tag{14.47}$$

By duality with respect to the complementation c, fuzzy dilation is then defined by:

$$\forall x \in \mathcal{S},\ \delta_\nu(\mu)(x) = \sup_{y\in\mathcal{S}} t[\nu(x-y),\mu(y)], \tag{14.48}$$

where t is the t-norm associated to the t-conorm T with respect to the complementation c . This definition of dilation corresponds to the following set equivalence:

$$x \in \delta_B(x) \Leftrightarrow \check{B}_x \cap X \neq \emptyset \Leftrightarrow \exists y \in \mathcal{S},\ y \in \check{B}_x \cap X. \tag{14.49}$$

Here, intersection $\cap$ has been translated in terms of a t-norm t and the existential symbol by a supremum.

This form of fuzzy dilation and fuzzy erosion are very general, and several definitions found in the literature appear as particular cases (such as Bandemer and Näther, 1992; de Baets and Kerre, 1993; Sinha and Dougherty, 1992; Rosenfeld, 1984; Kaufmann and Gupta, 1988; Goetcherian, 1980).

Finally, fuzzy opening (respectively fuzzy closing) is simply defined as the combination of a fuzzy erosion followed by a fuzzy dilation (respectively a fuzzy dilation followed by a fuzzy erosion), by using dual t-norms and t-conorms.

Weak t-norms and t-conorms are weaker forms of t-norms and t-conorms: they are not associative and do not admit 1 (respectively 0) as unit element, in general. If we replace t-norms and t-conorms by these weaker forms in the previous construction, then Eqs. (14.47,14.48) appear as a generalization of the definitions proposed in Sinha and Dougherty, 1993. But they lead to weaker properties, and are therefore somewhat less interesting from a morphological point of view.

Properties of basic fuzzy morphological operations. The detail of properties for various definitions can be found in Bloch and Maître, 1995. We summarize here the main properties when using t-norms and t-conorms:

- duality of erosion and dilation (respectively opening and closing) with respect to the complementation c;
- compatibility with classical morphology if the structuring element is binary;
- translation-invatiance (see Sec. 2);
- local knowledge property;
- continuity if the t-norm is continuous (which is most often the case);
- increasingness of all operations with respect to inclusion;
- extensivity of dilation and anti-extensivity of erosion iff $\nu(0) = 1$ (this corresponds to the condition that the origin should belong to the structuring element in the crisp case);
- extensivity of closing, anti-extensivity of opening and idempotence of these two operations iff $t[b, T(c(b), a)] \leq a$, which is satisfied for Lukasiewicz t-norm ($t(a, b) = \max(0, a+b-1)$) and t-conorm ($T(a, b) = \min(1, a+b)$);
- commutation of dilation with union (and of erosion with intersection);
- iteration property of dilation ($\delta_r \delta_s = \delta_{r+s}$) and of erosion.

Fuzzy morphology using adjunction and residual implications. A second type of approach is based on the notion of adjunction and fuzzy implications. Here the algebraic framework is the main guideline, which contrasts with the previous approach where duality was imposed in first place.

Fuzzy implication is often defined as (Dubois and Prade, 1991):

$$Imp(a, b) = T[c(a), b]. \tag{14.50}$$

Fuzzy inclusion is related to implication by the following equation:

$$\mathcal{I}(\nu, \mu) = \inf_{x \in \mathcal{S}} Imp[\nu(x), \mu(x)], \tag{14.51}$$

which allows to relate directly fuzzy erosion to fuzzy implication, leading to the general definition using t-conorm, and by duality also fuzzy dilation.

This suggests another way to define fuzzy erosion (and dilation), by using other forms of fuzzy implication. One interesting approach is to use residual implications:

$$Imp(a,b) = \sup\{\varepsilon \in [0,1], t(a,\varepsilon) \leq b\}. \tag{14.52}$$

This provides the following expression for the degree of inclusion:

$$\mathcal{I}(\nu,\mu) = \inf_{x\in\mathcal{S}} \sup\{\varepsilon \in [0,1], t(\nu(x),\varepsilon) \leq \mu(x)\}. \tag{14.53}$$

This definition coincides with the previous one for particular forms of t, typically Lukasiewicz t-norm.

The derivation of fuzzy morphological operators from residual implication has been proposed in de Baets, 1995, and then developed e.g. in de Baets, 1997 and Nachtegael and Kerre, 2000. One of its main advantages is that it leads to idempotent fuzzy closing and opening. This approach was formalized from the algebraic point of view of adjunction, as developed in Deng and Heijmans, 2002. This approach has then been used by other authors, such as in Maragos et al., 2001 and Maragos, 2005. This leads to general algebraic fuzzy erosion and dilation. Let us detail this approach. A fuzzy implication I is a mapping from $[0,1]\times[0,1]$ into $[0,1]$ which is decreasing in the first argument, increasing in the second one and satisfies $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$. A fuzzy conjunction is a mapping from $[0,1] \times [0,1]$ into $[0,1]$ which is increasing in both arguments and satisfies $C(0,0) = C(1,0) = C(0,1) = 0$ and $I(1,1) = 1$. If C is also associative and commutative, it is a t-norm. A pair of operators (I,C) are said adjoint if:

$$C(a,b) \leq c \Leftrightarrow b \leq I(a,c). \tag{14.54}$$

The adjoint of a conjunction is a residual implication.

Fuzzy dilation and erosion are then defined as:

$$\delta_\nu(\mu)(x) = \sup_y C(\nu(x-y),\mu(y)), \tag{14.55}$$

$$\varepsilon_\nu(\mu)(x) = \inf_y I(\nu(y-x),\mu(y)). \tag{14.56}$$

Note that (I,C) is an adjunction if and only if $(\varepsilon_\nu,\delta_\nu)$ is an adjunction.

Opening and closing derived from these operations by combination have all required properties, whatever the choice of C and I. Some properties of dilation and erosion, such as iterativity, require C and I to be associative and commutative.

If C is a t-norm, then the dilation is exactly the same as the one obtained in the first approach. To understand the relation between both approaches for erosion, we define $\hat{I}(a,b) = I(c(a),b)$ where c is a fuzzy complementation. In the following, we simply take $c(a) = 1-a$ which is the most usual complementation. Then $\hat{I}$ is increasing in both arguments, and if I is further assumed to be commutative and associative, $\hat{I}$ is a t-conorm. Equation (14.56) can be rewritten as: $\varepsilon_\nu(\mu)(x) = \inf_y \hat{I}(1-\nu(y-x), \mu(y))$, which corresponds to the fuzzy erosion of the first approach. The adjunction property can also be written as $C(a,b) \leq c \Leftrightarrow b \leq \hat{I}(1-a,c)$. However, pairs of dual t-norms and t-conorms are not identical to pairs of adjoint operators. Let us take a few examples. For $C = \min$, its adjoint is $I(a,b) = b$ if $b < a$, and 1 otherwise. But the derived $\hat{I}$ is the dual of the conjunction defined as $C(a,b) = 0$ if $b \leq 1-a$ and b otherwise. Conversely, the adjoint of this conjunction is $I(a,b) = \max(1-a,b)$, the dual of which is the minimum conjunction. Lukasiewicz operators are both adjoint and dual, which explains the exact correspondence between both approaches for these operators. Moreover, it can be proved that the condition for t-norms and t-conorms leading to idempotent opening and closing (i.e. $t(b, T(c(b),a) \leq a)$ is equivalent to the adjunction property between C and I for $t = C$ and $T = \hat{I}$. This new result completes the link between both approaches.

Fuzzy rough sets. We can now extend the links between rough sets and morphological operators derived in Sec. 3.2 to fuzzy rough sets. Using fuzzy mathematical morphology operators leads to fuzzy rough sets that have exactly the same properties as crisp rough sets, at least for particular t-norms and t-conorms (Bloch, 2000b). It turns out that these definitions using fuzzy erosion and dilation are generalizations of the ones proposed in Dubois and Prade, 1990, for $t = \min$ and $T = \max$ in a completely different context, using a fuzzy relation μ_R. The equivalence is obtained as in the crisp case by setting:

$$\mu_R(x,y) = \nu(y-x). \tag{14.57}$$

The interpretation is similar as in the crisp case: the degree of relation between x and y is equal to the degree to which $y-x$ belongs to the structuring element, i.e. to the degree to which y belongs to the structuring element translated at x. This approach has also been used in Nachtegael et al., 2000.

This extension brings together three different aspects of the information: rough sets represent coarseness, fuzzy sets represent vagueness and mathematical morphology brings a geometrical, topological and morphological aspect.

3.5 Spatial relationships and spatial reasoning from fuzzy mathematical morphology

Fuzzy distances derived from fuzzy dilation. The importance of distances in spatial reasoning is well established. Their extensions to fuzzy sets can be useful for dealing with imprecision and reasoning with semi-qualitative (or semi-quantitative) information. Several definitions can be found in the literature for distances between fuzzy sets (which is the main addressed problem). They can be roughly divided into two classes (see Bloch, 2003a for a review): distances that take only membership functions into account and that compare them pointwise, and distances that additionally include spatial distances. The definitions which combine spatial distance and fuzzy membership comparison allow for a more general analysis of structures in space, for applications where topological and spatial arrangement of the structures of interest is important (such as spatial reasoning).

Morphological dilations are a convenient tool to define distances in the second class (Bloch, 1999b). The relations described in Sec. 3.3 express distances in set theoretical terms, and are therefore easier to translate with nice properties than usual analytical expressions. We detail the examples of nearest point distance and Hausdorff distance.

Fuzzy nearest point distance. By translating Equation (14.33), we define a distance distribution (Rosenfeld, 1985) $\Delta_N(\mu,\mu')(n)$ that expresses the degree to which the distance between μ and μ' is less than n by:

(14.58) $$\Delta_N(\mu,\mu')(n) = f[\sup_{x\in\mathcal{S}} t[\mu(x), \delta_n(\mu')(x)], \sup_{x\in\mathcal{S}} t[\mu'(x), \delta_n(\mu)(x)]],$$

where δ_n is a fuzzy dilation of radius n ($\delta_n = (\delta_1)^n$), t is a t-norm, and f is a symmetrical function. The structuring element used in the dilatation δ_1 can simply be a unit ball, or a fuzzy set representing for instance the smallest sensitive unit in the image, along with the imprecision attached to it. In this case, it has to have a membership value equal to 1 at origin, in order to guarantee extensivity of dilations.

A distance density (Rosenfeld, 1985), i.e. a fuzzy number $D_N(\mu,\mu')(n)$ representing the degree to which the distance between μ and μ' is equal to n, can be obtained implicitly by:

(14.59) $$\Delta_N(\mu,\mu')(n) = \int_0^n D_N(\mu,\mu')(n')dn'.$$

Clearly, this expression is not very tractable and does not lead to a simple explicit expression of $D_N(\mu,\mu')(n)$. Therefore, we suggest to use an explicit method, exploiting the other morphological expressions if nearest point distance (see Sec. 3.3). The translation of these equivalences provides, for $n > 0$, the following distance density:

(14.60)
$$D_N(\mu,\mu')(n) = t[\sup_{x\in\mathcal{S}} t[\mu'(x), \delta_n(\mu)(x)], c[\sup_{x\in\mathcal{S}} t[\mu'(x), \delta_{(n-1)}(\mu)(x)]]]$$

or a symmetrical expression derived from this one, and:

$$D_N(\mu, \mu')(0) = \sup_{x \in \mathcal{S}} t[\mu(x), \mu'(x)]. \tag{14.61}$$

Fuzzy Hausdorff distance. From Equation (14.37), a distance distribution can be defined, by introducing fuzzy dilation:

$$\Delta_H(\mu, \mu')(n) = t[\inf_{x \in \mathcal{S}} T[\delta_n(\mu)(x), c(\mu'(x))], \inf_{x \in \mathcal{S}} T[\delta_n(\mu')(x), c(\mu(x))]], \tag{14.62}$$

where c is a complementation, t a t-norm and T a t-conorm. A distance density can be derived implicitly from this distance distribution.

A direct definition of a distance density can be obtained from the expression of $H_d(X, Y) = n$ (see Sec. 3.3). Translating this expression leads to a definition of the Hausdorff distance between two fuzzy sets μ and μ' as a fuzzy number:

$$H_d(\mu, \mu')(0) = t[\inf_{x \in \mathcal{S}} T[\mu(x), c(\mu'(x))], \inf_{x \in \mathcal{S}} T[\mu'(x), c(\mu(x))]], \tag{14.63}$$

$$\begin{aligned} H_d(\mu, \mu')(n) = t[&\inf_{x \in \mathcal{S}} T[\delta_n(\mu)(x), c(\mu'(x))], \inf_{x \in \mathcal{S}} T[\delta_n(\mu')(x), c(\mu(x))], \\ &T(\sup_{x \in \mathcal{S}} t[\mu(x), c(\delta_{(n-1)}(\mu')(x))], \sup_{x \in \mathcal{S}} t[\mu'(x), c(\delta_{(n-1)}(\mu)(x))])]. \end{aligned} \tag{14.64}$$

Properties. These definitions of fuzzy nearest point and Hausdorff distances (defined as fuzzy numbers) between two fuzzy sets do not necessarily share the same properties as their crisp equivalent. All distances are positive, in the sense that the defined fuzzy numbers have always a support included in $\mathbb{R}^+$. By construction, all defined distances are symmetrical with respect to μ and μ'. The separability property (i.e. $d(\mu, \nu) = 0 \Leftrightarrow \mu = \nu$) is not always satisfied. However, if μ is normalized (i.e. $\exists x, \mu(x) = 1$), we have for the nearest point distance $D_N(\mu, \mu)(0) = 1$ and $D_N(\mu, \mu)(n) = 0$ for $n > 1$. For the Hausdorff distance, $H_d(\mu, \mu')(0) = 1$ implies $\mu = \mu'$ for Lukasiewicz t-conorm, while it implies μ and μ' crisp and equal for $T = \max$. Also the triangular inequality is not satisfied in general.

Fuzzy adjacency from fuzzy dilation and set operations. Adjacency has a large interest in image processing, pattern recognition, spatial reasoning (Rosenfeld and Kak, 1976). A crisp definition of adjacency between crisp objects often leads to a low robustness, since the fact that two objects are adjacent or not may depend on one point only.

In order to account for possible errors or imprecisions, the framework of fuzzy sets is very useful. Two completely different ways for representing imprecision can be considered. In the first one, the satisfaction of the adjacency property between two objects is considered to be a matter of degree; this can be more appropriate than a binary index (Rosenfeld, 1979; Rosenfeld, 1984). The second one consists in introducing imprecision in the objects themselves, and to deal with fuzzy objects. Then obviously adjacency is also a matter of degree. Only the second way is addressed here. More details can be found in Bloch et al., 1997.

Adjacency is defined using fuzzy dilation, by translating Equation (14.40) into fuzzy terms. The degree of adjacency between μ and ν involving fuzzy dilation is then:

$$\mu_{adj}(\mu, \nu) = t[\mu_{\neg int}(\mu, \nu), \mu_{int}[\delta_{B_c}(\mu), \nu], \mu_{int}[\delta_{B_c}(\nu), \mu]]. \tag{14.65}$$

This definition represents a conjunctive combination of a degree of non-intersection $\mu_{\neg int}$ between μ and ν and a degree of intersection μ_{int} between one fuzzy set and the dilation of the other. The degree of intersection can be defined using a supremum of a t-norm (as for fuzzy dilation):

$$\mu_{int}(\mu, \nu) = \sup_{x \in \mathcal{S}} t[\mu(x), \nu(x)], \tag{14.66}$$

or using the normalized fuzzy surface (or volume) of $t(\mu, \nu)$. The degree of non-intersection is simply defined by $\mu_{\neg int} = 1 - \mu_{int}$. B_c can be taken as the elementary structuring element related to the considered connectivity, or as a fuzzy structuring element, representing for instance spatial imprecision (i.e. the possibility distribution of the location of each point).

This degree of adjacency (with any structuring element) is symmetrical, consistent with the binary case and decreases when the distance between both fuzzy sets increases.

Fuzzy directional relative position from conditional fuzzy dilation. Relationships between objects can be partly described in terms of relative position, like "to the left of". Since such concepts are rather ambiguous, although human beings have an intuitive and common way of understanding and interpreting them, they may find a better modeling in the framework of fuzzy sets as fuzzy relationships. This framework makes it possible to propose flexible definitions which fit the intuition and may include subjective aspects, depending on the application and on the requirements of the user. Almost all existing methods for defining fuzzy relative directional spatial position (see Bloch and Ralescu, 2003 for a review) rely on angle measurements between points of the two objects of interest (Krishnapuram et al., 1993; Miyajima and Ralescu, 1994; Keller and Wang, 1995; Matsakis and Wendling, 1999), and concern 2D objects (sometimes with possible extension to 3D).

Another approach was proposed in Bloch, 1999a. It is based on fuzzy dilation and consists of two steps:

1 We first define a fuzzy "landscape" around the reference object R as a fuzzy set such that the membership value of each point corresponds to the degree of satisfaction of the spatial relation under examination. This fuzzy region is defined by a fuzzy dilation of the reference object by a fuzzy structuring element expressing the direction of interest $\vec{u}$ along with its imprecision. For instance, the structuring element ν can be defined as:

$$\forall P \in \mathcal{S},\ \nu(P) = f(\arccos \frac{\vec{OP} \cdot \vec{u}}{\|\vec{OP}\|}), \quad \text{and} \quad \nu(O) = 1, \tag{14.67}$$

where O is the center of the structuring element and f is a decreasing function of $[0, \pi]$ into $[0, 1]$. An example is shown in Fig. 14.14. A fast algorithm for computing this fuzzy dilation is described in Bloch, 1999a.

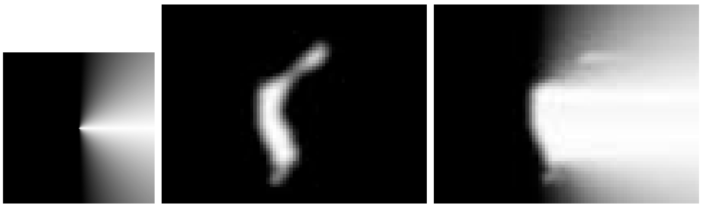


Figure 14.14. Fuzzy structuring element representing the relation "to the right of", a fuzzy reference object, and its dilation representing the region to the right of it (high grey values represent high membership values).

2 We then compare an object A to the fuzzy landscape attached to R, in order to evaluate how well this object matches with the areas having high membership values (i.e. areas that are in the desired direction). This is done using a fuzzy pattern matching approach (Dubois et al., 1988), which provides an evaluation as an interval instead of one number only. This makes a major difference with respect to all the previous approaches and, to our opinion, it provides a richer information about the considered relationship.

This definition is invariant with respect to translation, rotation and scaling, for 2D and 3D objects (crisp and fuzzy). When the distance between the objects increases, the objects are seen as points. The value of their relative position can be predicted only from the direction of interest and the direction in which one object goes far away from the reference object. Therefore the shape of the objects does no longer play any role in the assessment of their relative position. Finally, the behavior of the definition in cases where the reference object has strong concavities corresponds to what can be intuitively expected.

Reasoning on spatial relationships. Now, we address the second important issue in spatial reasoning. This includes fusion, since heterogeneous information has often to be combined in spatial reasoning, decision making and recognition (with a special focus on model-based recognition). Inference and logical reasoning are addressed in Sec. 4.

Fusion. Spatial reasoning aiming for instance at recognizing structures in an image has to deal with the combination of knowledge and information represented and modeled as described above. Usually, to achieve recognition, several spatial relationships to one or several spatial entities have to be combined, as well as information extracted from the image itself. For this combination step, the advantages of fuzzy sets lie in the variety of combination operators, which may deal with heterogeneous information expressed in a semi-quantitative framework (Dubois and Prade, 1985; Yager, 1991; Dubois et al., 1999). A classification of these operators with respect to their behavior (in terms of conjunctive, disjunctive, and compromise), the possible control of this behavior, their properties and their decisiveness proved to be useful for choosing an operator (Bloch, 1996).

Let us give a few examples. If we have different constraints about an object (for instance concerning the relations it should have with respect to another object) which have all to be satisfied, these constraints can be combined using a t-norm (a conjunction). This is typically the case when an object is described using relations to several objects or several relations of different types to the same object. If one object has to satisfy one relation or another one then a disjunction represented by a t-conorm has to be used. This occurs for instance when two symmetrical structures with respect to the reference object can be found. Mean operators can be used to combine several estimations and try to find a compromise between them.

Decision making and recognition. Let us now consider the introduction of fusion in model-based recognition procedures. We summarize here two distinct approaches. A first recognition approach, called global, uses the first type of question (1) raised at the beginning of Sec. 3.3 (define the degree to which a relation is satisfied between to given objects). The idea is to represent all available knowledge about the objects to be recognized. A typical example consists of graph-based representations. The model is then represented as a graph where nodes are objects and edges represent links between these objects. Both nodes and edges are attributed. Node attributes are characteristics of the objects, while edge attributes quantify spatial relationships between the objects. A data graph is then constructed from each image where the recognition has to be performed. Each region of the image (obtained after some processing) constitutes a node of this data graph, and edges represent spatial relationships between regions, as for the model graph. The comparison between representations is performed through the computation of similarities between model graph attributes and data graph attributes. The fusion takes mainly place at this level, in order to combine the similarity values for different relationships. The fusion results constitute an objective function to be optimized by a matching procedure. This approach can benefit from the huge literature on fuzzy comparison tools (see e.g. Bouchon-Meunier et al., 1996) and from recent developments on fuzzy morphisms (Perchant and Bloch, 2002). It has been used in facial feature recognition based on a rough model of a face (Cesar et al., 2002) and brain structure recognition based on an anatomical atlas (Perchant et al., 1999; Bengoetxea et al., 2002).

A second type of approach relies on the second type of question (2) raised at the beginning of Sec. 3.3 (define the area of space in which a relation to a given reference object is satisfied), and is called here progressive. In such a progressive approach, objects are recognized sequentially and their recognition makes use of knowledge about their relations with respect to other objects. Relations with respect to previously obtained objects can be combined at two different levels of the procedure. First, fusion can occur in the spatial domain, using spatial fuzzy sets (Bloch et al., 2003). The result of this fusion allows to build a fuzzy region of interest in which the search of a new object will take place, in a process similar to focalization of attention. In a sequential procedure, the amount of available spatial relations increases with the number of processed objects. Therefore, the recognition of the most difficult structures, usually treated in the last steps, will be focused in a more restricted area. This approach has been used in medical imaging (Bloch et al., 2003; Colliot et al., 2004), as well as in mobile robotics to reason about the spatial position of the robot and the structure of its environment (Bloch and Saffiotti, 2002). Another fusion level occurs during the final decision step, i.e. segmentation and recognition of a structure. For this purpose, it was suggested in Colliot et al., 2004 to introduce relations in the evolution scheme of a deformable model, in which they are fused with other types of numerical information, usually edge and regularity constraints.

4. Logics

In this section, we explain how mathematical morphology relates to logics of space. As seen in Sec. 3.3, mathematical morphology can be considered as a spatial reasoning tool, with its two components: spatial knowledge representation and reasoning. Now we go one step further about the reasoning aspect, and we show how morphological operators can be applied on logical formulas (Sec. 4.1), and can be used to define a modal logic (Sec. 4.2). This leads to qualitative representation of spatial relationships (Sec. 4.3), thus enhancing the power of logical reasoning with morphological aspects.

4.1 Morphology and propositional logics

We first consider the framework of propositional logics (note that this subsection is partly reproduced from Bloch and Lang, 2000).

In the knowledge representation community, propositional formulas are used to encode either pieces of knowledge (which may be generic – for instance, integrity constraints – or factual) or "preference items" (such as opinions, desires or goals), and are then used for complex reasoning or decision making tasks. These tasks often make use of operations on propositional formulas which are very similar to those considered in mathematical morphology. We give a (non-exhaustive) list of examples:

- *belief revision* (as shown by Katsuno and Mendelzon, 1991) consists of the following operation: let φ and ψ be two propositional formulas. The models of the revision $\varphi \circ \psi$ of φ by ψ are the models of ψ which are closest (with respect to a given distance) to a model of φ. Intuitively, using the language of mathematical morphology, it means that φ has to be dilated enough to intersect with some models of ψ. *Belief update* (Katsuno and Mendelzon, 1991) proceeds to the same kind of dilation but on each individual model of φ and then takes the union of all obtained sets of models.

- *belief merging* (Konieczny and Pino-Pérez, 1998) consists in finding the best compromise between a finite set of formulas φ_1, ..., φ_n, which amounts to selecting the models which minimize the aggregation (using some given operator) of the distances to each of the φ's. This amounts intuitively to dilate simultaneously all the φ's until they intersect. Similar operations are at work for the aggregation of preferences in group decision making as proposed in Lafage and Lang, 2000.

- one of the tasks involved in *similarity-based reasoning* (Esteva et al., 1997; Dubois et al., 1997) consists in determining if a formula φ approximatively entails a formula ψ by looking to what extent ψ has to be extended so as to contain all models of φ, which again corresponds to a dilation (and to directed Hausdorff distance).

- *reasoning with supermodels* (Ginsberg et al., 1998) uses models of a formula φ which are robust enough to resist some perturbations. In some cases, obtaining supermodels consists in eroding the formula so as to be far enough from the countermodels of φ. Again this corresponds to a classical operation of mathematical morphology (erosion). Another close notion, evoked by Lafage and Lang, 2000, is the search for the most representative worlds of a formula.

- in *abductive reasoning* (Pino-Pérez and Uzcátegui, 1999), preferred explanations of a formula are defined based on a set of axioms, several of which being close to properties of morphological operators. Erosion appears as a useful tool in this context (Bloch et al., 2001; Bloch et al., 2004).

In this section, we investigate how and why mathematical morphology can be applied on logical formulas. First we note that the fact that a propositional formula can be equivalently defined by the set of its models enables us to apply easily all (set-theoretic) definitions of mathematical morphology to logical objects (worlds, formulas). This will lead us not only to rewriting well-known logical operations used for reasoning or decision making, but also to designing new kinds of logical objects or notions by transposing basic morphological operations to propositional logic. One may view morphological operators as transformations applied on formulas, leading to reasoning or decision making tools.

Basic logical concepts. Let PS be a finite set of propositional symbols. The set of formulas (generated by PS and the usual connectives) is denoted by Φ. Well-formed formulas will be denoted by Greek letters φ, ψ... Interpretations will be denoted by ω, ω' and the set of all interpretations for Φ by Ω. $Mod(\varphi) = \{\omega \in \Omega \mid \omega \models \varphi\}$ is the set of all models of φ (i.e. all interpretations for which φ is true).

The underlying idea for constructing morphological operations on logical formulas is to consider formulas and interpretations from a set theoretical perspective. Since Φ is isomorphic to 2^{Ω}, i.e., knowing a formula is equivalent to knowing the set of its models (and conversely, any set of models corresponds to a formula), we can identify φ with the set of its models $Mod(\varphi)$, and then apply set-theoretic morphological operations. We recall that $Mod(\varphi \vee \psi) = Mod(\varphi) \cup Mod(\psi)$, $Mod(\varphi \wedge \psi) = Mod(\varphi) \cap Mod(\psi)$, $Mod(\varphi) \subseteq Mod(\psi)$ iff $\varphi \models \psi$, and φ is consistent iff $Mod(\varphi) \neq \emptyset$.

Dilation and erosion of a formula. Using the previous equivalences, we propose to define dilation and erosion of a formula as follows:

$$Mod(\delta_B(\varphi)) = \{\omega \in \Omega \mid \check{B}_\omega \wedge \varphi \ consistent\}, \tag{14.68}$$

$$Mod(\varepsilon_B(\varphi)) = \{\omega \in \Omega \mid B_\omega \models \varphi\}. \tag{14.69}$$

In these equations, the structuring element B represents a relation between worlds, i.e. $\omega' \in B_\omega$ iff ω' satisfies some relationship with ω, and $\check{B}_\omega$ is defined by $\omega' \in \check{B}_\omega \Leftrightarrow \omega \in B_{\omega'}$. The condition in Equation (14.68) expresses that the set of worlds in relation to ω should be consistent with φ. The condition in Equation (14.69) expresses that all worlds in relation to ω should be models of φ.

Structuring element. There are several possible ways to define structuring elements or more generally binary relations between worlds in a context of formulas. We suggest here a few ones. The relation between worlds defines a "neighbourhood" of worlds (equivalent to the neighbourhood function in Sec. 2.3). If it is symmetrical, it leads to symmetrical structuring elements. If it is reflexive, it leads to structuring elements such that $\omega \in B_\omega$, which leads to interesting properties, as will be seen later. For instance, this relationship can be an accessibility relation as in normal modal logics (Hughes and Cresswell, 1968). An interesting way to choose the relationship is to base it on distances between worlds. This allows to define sequences of increasing structuring elements defined as the balls of a distance. From any distance d between worlds, a distance from a world to a formula is derived as a distance from a point to a set: $d_N(\omega, \varphi) = \min_{\omega' \models \varphi} d(\omega, \omega')$.

The most commonly used distance between worlds in knowledge representation—especially in belief revision (Dalal, 1988), belief update (Katsuno and Mendelzon, 1991), merging (Konieczny and Pino-Pérez, 1998) or preference representation (Lafage and Lang, 2000)—is the Hamming distance d_H where $d_H(\omega, \omega')$ is the number of propositional symbols that are instantiated differently in both worlds. By default, we take d to be d_H.

Then dilation and erosion of size n are defined from Eqs. (14.68,14.69) by using the distance balls of radius n as structuring elements:

$$\begin{aligned} Mod(\delta_n(\varphi)) &= \{\omega \mid \exists \omega', \omega' \models \varphi \ and \ d_H(\omega, \omega') \leq n\} \\ &= \{\omega \mid d_N(\omega, \varphi) \leq n\}, \end{aligned} \tag{14.70}$$

$$\begin{aligned} Mod(\varepsilon_n(\varphi)) &= \{\omega \mid \forall \omega', d_H(\omega, \omega') \leq n \Rightarrow \omega' \models \varphi\} \\ &= \{\omega \mid d_N(\omega, \neg\varphi) > n\}. \end{aligned} \tag{14.71}$$

From operations with the unit ball we define the external (respectively internal) boundary of φ as $\delta_1(\varphi) \wedge \neg\varphi$ (respectively $\varphi \wedge \neg\varepsilon_1(\varphi)$), corresponding to the worlds that are exactly at distance 1 of φ (resp. of $\neg\varphi$).

Properties. The main properties of dilation and erosion, which are satisfied in mathematical morphology on sets, hold also in the logical setting proposed here, since the algebraic frameworks are the same up to an isomorphism.

Monotonicity: Both operators are increasing with respect to φ, i.e. if $\varphi \models \psi$, then $\delta_B(\varphi) \models \delta_B(\psi)$ and $\varepsilon_B(\varphi) \models \varepsilon_B(\psi)$, for any relation B. Dilation is increasing with respect to the relation, while erosion is decreasing, i.e. if $\forall\omega \in \Omega, B_\omega \subseteq B'_\omega$, then $\delta_B(\varphi) \models \delta_{B'}(\varphi)$ and $\varepsilon_{B'}(\varphi) \models \varepsilon_B(\varphi)$.

Extensivity: Dilation is extensive ($\varphi \models \delta_B(\varphi)$) if B is derived from a reflexive relation (as is the case for distance based dilation, since if $\omega \models \varphi$, then $d_N(\omega, \varphi) = 0$), and erosion is anti-extensive ($\varepsilon_B(\varphi) \models \varphi$) under the same conditions.

Iteration: Dilation and erosion satisfy an iteration property. For instance for distance based operations, we have:

$$\delta_{n+n'}(\varphi) = \delta_{n'}[\delta_n(\varphi)] = \delta_n[\delta_{n'}(\varphi)],$$

$$\varepsilon_{n+n'}(\varphi) = \varepsilon_{n'}[\varepsilon_n(\varphi)] = \varepsilon_n[\varepsilon_{n'}(\varphi)].$$

Commutativity with union or intersection: Dilation commutes with union or disjunction: for any family $\varphi_1, ...\varphi_m$ of formulas, we have: $\delta_B\left(\bigvee_{i=1}^m \varphi_i\right) = \bigvee_{i=1}^m \delta_B(\varphi_i)$. Erosion on the other hand commutes with intersection or conjunction.

In general dilation (resp. erosion) does not commute with intersection (resp. union), and only an inclusion relation holds: $\delta_B(\varphi \wedge \psi) \models \delta_B(\varphi) \wedge \delta_B(\psi)$.

Adjunction relation: $(\varepsilon_B, \delta_B)$ is an adjunction (moreover, if two operators form an adjunction, they are an erosion and a dilation respectively), i.e. $\delta_B(\psi) \models \varphi$ iff $\psi \models \varepsilon_B(\varphi)$.

Duality: Dilation and erosion (respectively opening and closing) are dual operators with respect to the negation: $\varepsilon_B(\varphi) = \neg\delta_{\check{B}}(\neg\varphi)$ which allows to deduce properties of an operator from those of its dual operator.

Relations to distances: Equation (14.70) is an example of how to derive a dilation from a distance. Conversely, we have: $d_N(\omega, \varphi) = \min\{n \in \mathbb{N} \mid \omega \models \delta_n(\varphi)\}$. Distances between formulas can also be derived from dilation, as minimum distance and Hausdorff distance. For instance the minimum distance (i.e. nearest world distance) is expressed as: $d_N(\varphi, \psi) = \min_{\omega \models \varphi, \omega' \models \psi} d_H(\omega, \omega') = \min\{n \in \mathbb{N} \mid \delta_n(\varphi) \wedge \psi \neq \emptyset \; and \; \delta_n(\psi) \wedge \varphi \neq \emptyset\}$. This means that the minimum distance is attained for the minimum size of dilation of each formula such that it becomes consistent with the other.

Opening and closing are defined classically by composition and have the same properties as the corresponding operators on sets. Filters, as described in Sec. 2.4, can be applied on formulas as well, for instance for approximation and simplification purposes.

All these definitions and properties allow us to formalize problems of fusion, revision, abduction mentioned at the beginning of this subsection in morphological terms.

4.2 Morphological modal logic

When looking at the algebraic properties of mathematical morphology operators on the one hand, and of modal logic operators on the other hand, several similarities can be shown, and suggest that links between both theories are worth to be investigated. A pair of modal operators $(\Box, \Diamond)$ is defined in Bloch, 2002, as morphological erosion and dilation. This section summarizes this approach.

Until now mathematical morphology has been used mainly for quantitative and semi-quantitative (or semi-qualitative) representations of spatial relations. For qualitative spatial reasoning, several symbolic approaches have been developed, but mathematical morphology has not been widely used in this context. In Bloch, 2002, it was shown how modal operators based on morphological operators can be used for symbolic representations of spatial relations.

In a similar way as in Jeansoulin and Mathieu, 1994, the modal operators are used here for representing spatial relationships, and classical predicates represent the semantic part of the information. While inclusion and adjacency are considered in Jeansoulin and Mathieu, 1994, we consider here more spatial relationships, including metric ones, and model all of them using mathematical morphology.

Lattice structure. We use the same notations as in Sec. 4.1. We use standard Kripke's semantics and denote by $\mathcal{M}$ a model composed of a set of worlds Ω, a binary relation R between worlds and a truth valuation. Considering the inclusion relation on 2^Ω, $(2^\Omega, \subseteq)$ is a Boolean complete lattice. Similarly a lattice (which is isomorphic to 2^Ω) is defined on $\Phi_\equiv$, where $\Phi_\equiv$ denotes the quotient space of Φ by the equivalence relation between formulas (with the equivalence defined as $\varphi \equiv \psi$ iff $Mod(\varphi) = Mod(\psi)$). In the following, this is implicitly assumed, and we simply use the notation Φ. Any subset $\{\varphi_i\}$ of Φ has a supremum $\bigvee_i \varphi_i$, and an infimum $\bigwedge_i \varphi_i$ (corresponding respectively to union and intersection in 2^Ω). The greatest element is $\top$ and the smallest one is $\perp$ (corresponding respectively to 2^Ω and $\emptyset$). This lattice structure is important from the algebraic point of view of mathematical morphology. Indeed, it is the fundamental structure on which adjunctions and morphological operators can be defined.

A canonical formula φ_ω associated with a world ω is defined by:

(14.72) $$Mod(\varphi_\omega) = \{\omega\}.$$

Let $\mathcal{C}$ be the subset of Φ containing all canonical formulas. The canonical formulas are sup-generating, i.e:

$$\forall \varphi \in \Phi,\ \exists \{\varphi_i\} \subseteq \mathcal{C},\ \varphi \equiv \bigvee_i \varphi_i. \tag{14.73}$$

The formulas φ_i are associated with the worlds ω_i which satisfy φ: for all ω_i such that $\omega_i \models \varphi$, $\varphi_i \equiv \varphi_{\omega_i}$. This decomposition is useful for some proofs.

Neighborhood function (or structuring element) as accessibility relation. The structuring element B representing a relationship between worlds defines a "neighbourhood" of worlds. This corresponds to the notion of neighbourhood function of Sec. 2.3. We propose to define this relationship as an accessibility relation as in normal modal logics (Hughes and Cresswell, 1968; Chellas, 1980).

An interesting way to choose the relationship is to base it on distances between worlds, as mentioned in Sec. 4.1. Another way to choose the relationship is to rely on an indistinguishability relation between worlds (Orlowska, 1993; Balbiani and Orlowska, 1999), for instance based on spatial attributes of spatial entities represented by these worlds. Interestingly enough, as shown in Orlowska, 1993, modal logics based on such relationships show some links with Pawlak's work on rough sets and rough logic (Pawlak, 1982; Pawlak, 1987), while rough sets can be constructed from morphological operators as shown in Bloch, 2000b. Also, the modal logic based on rough sets described e.g. in Yao and Lin, 1996, has links with the morphological modal logic described below. An accessibility relation can be defined from any neighbourhood function B as follows:

$$R(\omega, \omega') \text{ iff } \omega' \in B(\omega). \tag{14.74}$$

Conversely, a neighbourhood function can be defined from an accessibility relation using this equivalence. This is similar to the notions of Sec. 2.3.

The accessibility relation R is reflexive iff $\forall \omega \in \Omega,\ \omega \in B(\omega)$. It is symmetrical iff $\forall (\omega, \omega') \in \Omega^2,\ \omega \in B(\omega')$ iff $\omega' \in B(\omega)$. In general, accessibility relations derived from a neighbourhood function are not transitive. Indeed in general if $\omega' \in B(\omega)$ and $\omega'' \in B(\omega')$, we do not necessarily have $\omega'' \in B(\omega)$.

Modal logic from morphological dilations and erosions. Modal operators $\Box$ (necessity) and $\Diamond$ (possibility) are usually defined from an accessibility relation (Chellas, 1980) as:

$$\mathcal{M}, \omega \models \Box\varphi \text{ iff } \forall \omega' \in \Omega \text{ such that } R(\omega, \omega'), \mathcal{M}, \omega' \models \varphi, \tag{14.75}$$

$$\mathcal{M}, \omega \models \Diamond\varphi \text{ iff } \exists \omega' \in \Omega,\ R(\omega, \omega') \text{ and } \mathcal{M}, \omega' \models \varphi, \tag{14.76}$$

where $\mathcal{M}$ is a standard model related to R, that we will omit in the following in order to simplify notations (it will be always implicitly related to the considered accessibility relation).

Equation (14.75) can be rewritten as:

$$\begin{aligned} \omega \models \Box\varphi &\Leftrightarrow \{\omega' \in \Omega \mid R(\omega, \omega')\} \models \varphi \\ &\Leftrightarrow \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \models \varphi \\ &\Leftrightarrow B(\omega) \models \varphi, \end{aligned}$$

which corresponds exactly to the definition of the erosion of a formula as defined in Equation (14.69).

Similarly, Equation (14.76) can be rewritten as:

$$\begin{aligned} \omega \models \Diamond\varphi &\Leftrightarrow \{\omega' \in \Omega \mid R(\omega, \omega')\} \cap Mod(\varphi) \neq \emptyset \\ &\Leftrightarrow \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \cap Mod(\varphi) \neq \emptyset \\ &\Leftrightarrow B(\omega) \cap Mod(\varphi) \neq \emptyset, \end{aligned}$$

which exactly corresponds to a dilation according to Equation (14.68).

This shows that we can define modal operators based on an accessibility relation as erosion and dilation with a neighbourhood function:

$$\Box\varphi \equiv \varepsilon_B(\varphi), \tag{14.77}$$

$$\Diamond\varphi \equiv \delta_{\check{B}}(\varphi). \tag{14.78}$$

Let us now list the main properties of these operators. All results below can be found in Bloch, 2002, along with the corresponding proofs. Note that some results are direct consequences of the results of Sec. 2.3. For instance, **T** is deduced from Proposition 14.21, from which **5c** and **4c** are derived.

LEMMA 14.30 *The modal logic built from morphological erosions and dilations has the following theorems and rules of inference (we use similar notations as in Chellas, 1980):*

- **T**: $\Box\varphi \to \varphi$ *and* $\varphi \to \Diamond\varphi$ *iff* $\forall\omega \in \Omega, \omega \in B(\omega)$ *(reflexive accessibility relation).*
- **Df**: $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ *and* $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$.
- **D**: $\Box\varphi \to \Diamond\varphi$ *iff* R *is serial (or in other words,* $\forall\omega \in \Omega, B(\omega) \neq \emptyset$*).*
- **B**: $\Diamond\Box\varphi \to \varphi$ *and* $\varphi \to \Box\Diamond\varphi$ *for symmetrical* B.
- **5c**: $\Box\Diamond\varphi \to \Diamond\varphi$ *and* $\Box\varphi \to \Diamond\Box\varphi$ *iff* $\forall\omega \in \Omega, \omega \in B(\omega)$.
- **4c**: $\Box\Box\varphi \to \Box\varphi$ *and* $\Diamond\varphi \to \Diamond\Diamond\varphi$ *iff* $\forall\omega \in \Omega, \omega \in B(\omega)$.

- **N**: $\Box\top$ *and* $\neg\Diamond\perp$.
- **M**: $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$ *and* $(\Diamond\varphi \vee \Diamond\psi) \rightarrow \Diamond(\varphi \vee \psi)$.
- **M'**: $\Diamond(\varphi \wedge \psi) \rightarrow (\Diamond\varphi \wedge \Diamond\psi)$ *and* $(\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$.
- **C**: $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ *and* $\Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi)$.
- **R**: $(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi)$ *and* $\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$.
- **RN**:
$$\frac{\varphi}{\Box\varphi}.$$
- **RM**:
$$\frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi} \; and \; \frac{\varphi \rightarrow \psi}{\Diamond\varphi \rightarrow \Diamond\psi}.$$
- **RR**:
$$\frac{(\varphi \wedge \varphi') \rightarrow \psi}{(\Box\varphi \wedge \Box\varphi') \rightarrow \Box\psi} \; and \; \frac{(\varphi \vee \varphi') \rightarrow \psi}{(\Diamond\varphi \vee \Diamond\varphi') \rightarrow \Diamond\psi}.$$
- **RE**:
$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi} \; and \; \frac{\varphi \leftrightarrow \psi}{\Diamond\varphi \leftrightarrow \Diamond\psi}.$$
- **K**: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ *and by duality* $(\neg\Diamond\varphi \wedge \Diamond\psi) \rightarrow \Diamond(\neg\varphi \wedge \psi)$.

Since the proposed system contains **Df**, **N**, **C** and is closed by **RM**, it is a normal modal logic (Chellas, 1980).

LEMMA 14.31 *On the contrary, the following expressions are not satisfied in general:*

- **5**: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ *(since for a symmetrical B the dilation followed by an erosion is a closing which does not necessarily contains the dilation).*
- **4**: $\Box\varphi \rightarrow \Box\Box\varphi$ *(since eroding a region twice produces a smaller region if $\omega \in B(\omega)$).*

Let us now denote by $\Box^n$ the iteration of n times $\Box$ (i.e. n erosions by the same structuring element). Since the succession of n erosions by a structuring element is equivalent to one erosion by a larger structuring element, of size n (iterativity property of erosion), $\Box^n$ is a new modal operator, constructed as in Equation (14.77). In a similar way, we denote by $\Diamond^n$ the iteration of n times $\Diamond$, which is again a new modal operator, due to iterativity property of dilation, constructed as in Equation (14.78) with a structuring element or neighbourhood of size n. We set $\Box^1 = \Box$ and $\Diamond^1 = \Diamond$.

We also have the following theorems for symmetrical B and $\omega \in B(\omega)$:

- $\Box^n \Box^{n'} \varphi \leftrightarrow \Box^{n+n'} \varphi$, and $\Diamond^n \Diamond^{n'} \varphi \leftrightarrow \Diamond^{n+n'} \varphi$ (iterativity properties of dilation and erosion). This property holds also in a more general case, without assumption on the symmetry of B.
- $\Diamond\Box\Diamond\Box\varphi \leftrightarrow \Diamond\Box\varphi$, and $\Box\Diamond\Box\Diamond\varphi \leftrightarrow \Box\Diamond\varphi$ (idempotence of opening and closing). This is actually a theorem from any KB logic: $\Diamond\Box\Diamond\Box\varphi \rightarrow \Diamond\Box\varphi$ is **B** applied to $\Diamond\Box\varphi$ and $\Diamond\Box\varphi \rightarrow \Diamond\Box\Diamond\Box\varphi$ comes from **B** applied to $\Box\varphi$ and from **RM**.
- $\Diamond\Box\Diamond\varphi \leftrightarrow \Diamond\varphi$, and $\Box\Diamond\Box\varphi \leftrightarrow \Box\varphi$.
- More generally, we derive from properties of opening and closing the following theorems:

  $$\Diamond^n \Box^n \Diamond^{n'} \Box^{n'} \varphi \leftrightarrow \Diamond^{\max(n,n')} \Box^{\max(n,n')} \varphi,$$

  and

  $$\Box^n \Diamond^n \Box^{n'} \Diamond^{n'} \varphi \leftrightarrow \Box^{\max(n,n')} \Diamond^{\max(n,n')} \varphi.$$

  See also the paragraph on granulometries in Sec. 2.4.
- For $n < n'$, the following expressions are theorems (if R is reflexive): $\Diamond^n \varphi \rightarrow \Diamond^{n'} \varphi$, $\Box^{n'} \varphi \rightarrow \Box^n \varphi$, $\Box^n \Diamond^n \varphi \rightarrow \Box^{n'} \Diamond^{n'} \varphi$, $\Diamond^{n'} \Box^{n'} \varphi \rightarrow \Diamond^n \Box^n \varphi$.

Modal operators from adjunction. Now, we consider the more general framework of algebraic erosions and dilations and the fundamental properties of adjunction (Heijmans and Ronse, 1990; Heijmans, 1994, Sec. 2.3).

Generalizing the definitions of Bloch and Lang, 2000, an algebraic dilation δ on Φ is defined as an operation which commutes with disjunction, and an algebraic erosion ε as an operation which commutes with conjunction, i.e. we have the two following expressions for any family $\{\varphi_i\}$:

(14.79) $$\delta\Big(\bigvee_i \varphi_i\Big) \equiv \bigvee_i \delta(\varphi_i),$$

(14.80) $$\varepsilon\Big(\bigwedge_i \varphi_i\Big) \equiv \bigwedge_i \varepsilon(\varphi_i).$$

One of the fundamental concept in the algebraic framework is the one of adjunction (see Sec. 2.3). Using similar concepts modal operators can be defined on Φ. A pair of modal operators $(\Box, \Diamond')$ is an adjunction on Φ iff:

(14.81) $$\forall(\varphi,\psi) \in \Phi^2,\ \models (\Diamond'\varphi \rightarrow \psi \ \equiv\ \varphi \rightarrow \Box\psi),$$

or in other words:

$$\frac{\varphi \rightarrow \Box\psi}{\Diamond'\varphi \rightarrow \psi} \text{ and } \frac{\Diamond'\varphi \rightarrow \psi}{\varphi \rightarrow \Box\psi}.$$

In terms of worlds, this can also be expressed as:

$$\forall(\varphi, \psi) \in \Phi^2,\ Mod(\Diamond'\varphi) \subseteq Mod(\psi) \text{ iff } Mod(\varphi) \subseteq Mod(\Box\psi). \tag{14.82}$$

At this point, we use the notation $(\Box, \Diamond')$ instead of the classical notation $(\Box, \Diamond)$ because, as will be seen later, the two operators are not necessarily dual. In general they are two different modal operators.

LEMMA 14.32 *If $(\Box, \Diamond')$ is an adjunction on Φ, then $\Box$ is an algebraic erosion, and $\Diamond'$ is an algebraic dilation, i.e. for any family $\{\varphi_i\}$, we have:*

$$\Box \bigwedge_i \varphi_i \equiv \bigwedge_i \Box\varphi_i, \tag{14.83}$$

$$\Diamond' \bigvee_i \varphi_i \equiv \bigvee_i \Diamond'\varphi_i. \tag{14.84}$$

These equivalences are also true for empty families, since we have $\Box\top \equiv \top$ and $\Diamond'\bot \equiv \bot$.

LEMMA 14.33 *Let $(\Box, \Diamond')$ be an adjunction on the lattice of logical formulas. The modal logic based on these operators has the following theorems and rules of inference (we use similar notations as in Theorem 14.30 but $\Diamond$ has to be replaced by $\Diamond'$):* **B**, **N**, **M**, **M'**, **C**, **R**, **RN**, **RM**, **RR**, **RE**, **K**.

The proof is derived mainly from Theorem 14.32, from Eqs. (14.73,14.81–14.84) and from the following result:

LEMMA 14.34 *We can write $\Box$ and $\Diamond'$ as:*

$$\Box\varphi \equiv \bigvee\{\psi \in \Phi,\ \Diamond'\psi \rightarrow \varphi\}, \tag{14.85}$$

$$\Diamond'\varphi \equiv \bigwedge\{\psi \in \Phi,\ \varphi \rightarrow \Box\psi\}. \tag{14.86}$$

Again formulas are considered up to the equivalence relation, and therefore $\bigvee$ and $\bigwedge$ are taken over a finite family.

LEMMA 14.35 **T**, **5c** *and* **4c** *are not always satisfied, and we have the following results:*

- **T** *iff* $\forall\omega \in \Omega,\ \omega \models \Diamond'\varphi_\omega$,

- **5c** *iff* $\forall \omega \in \Omega,\ \omega \models \Diamond' \varphi_\omega$,
- **4c** *iff* $\forall \omega \in \Omega,\ \omega \models \Diamond' \varphi_\omega$.

Note that the condition on B for **T** in Theorem 14.30 corresponds to the one above, and we have $B(\omega) = Mod(\Diamond \varphi_\omega)$.

LEMMA 14.36 *We have the two following additional theorems:*

- $\Box\Diamond'\Box\varphi \leftrightarrow \Box\varphi$ *and* $\Diamond'\Box\Diamond'\varphi \leftrightarrow \Diamond'\varphi$.
- $\Diamond'\Box\Diamond'\Box\varphi \leftrightarrow \Diamond'\Box\varphi$ *and* $\Box\Diamond'\Box\Diamond'\varphi \leftrightarrow \Box\Diamond'\varphi$.

LEMMA 14.37 *Let* $(\Box, \Diamond')$ *be an adjunction on* Φ. *Let* $\Box_* \varphi \equiv \neg\Box\neg\varphi$ *and* $\Diamond'_* \varphi \equiv \neg\Diamond'\neg\varphi$. *Then* $(\Diamond'_*, \Box_*)$ *is an adjunction.*

This property expresses a kind of duality between both operators.

Note that we do not always have: **Df**: $\Diamond'\varphi \leftrightarrow \neg\Box\neg\varphi$ and $\Box\varphi \leftrightarrow \neg\Diamond'\neg\varphi$, nor **D**: $\Box\varphi \rightarrow \Diamond'\varphi$.

LEMMA 14.38 **Df** *is satisfied by an adjunction* $(\Box, \Diamond')$ *if and only if* $\Diamond'$ *satisfies the following property:*

$$\forall (\omega, \omega') \in \Omega^2,\ \omega \models \Diamond'\varphi_{\omega'} \text{ iff } \omega' \models \Diamond'\varphi_\omega. \tag{14.87}$$

D *is satisfied by an adjunction* $(\Box, \Diamond')$ *if* $\Diamond'$ *satisfies one of the two following properties:*

$$\forall \omega \in \Omega,\ \omega \models \Diamond'\varphi_\omega \tag{14.88}$$

or

(14.89)

$$\forall (\omega, \omega') \in \Omega^2,\ \omega \models \Diamond'\varphi_{\omega'} \text{ iff } \omega' \models \Diamond'\varphi_\omega \text{ and } \{\omega', \omega' \models \Diamond'\varphi_\omega\} \neq \emptyset.$$

The last result (see Proposition 14.20) means in particular that we can have **D** without having **T**.

In cases where **Df** is satisfied, then we note simply $\Diamond$ instead of $\Diamond'$.

LEMMA 14.39 *The operators* $(\Box, \Diamond)$ *defined by Eqs. (14.77,14.78) build an adjunction in the case* B *is symmetrical.*

This shows that modal operators derived from morphological erosions and dilations are particular cases of modal operators derived from algebraic erosions and dilations.

All these results show that the use of general algebraic dilations and erosions defined from the adjunction property lead to the properties of normal modal logics. This justifies the use of Kripke's semantics. This also guarantees a completeness result.

Characterizing modal logics in terms of morphological operators. Conversely, the following result shows that modal operators satisfying some axioms can be expressed in morphological terms.

LEMMA 14.40 *If two modal operators $\Box$ and $\Diamond$ satisfy* **B** *and* **RM**, *then* $(\Box, \Diamond)$ *is an adjunction on* Φ, $\Box$ *is an algebraic erosion and* $\Diamond$ *is an algebraic dilation.*

Moreover, if we define a relation R between worlds by $R(\omega, \omega')$ iff $\omega \models \Diamond\varphi_{\omega'}$, where φ_ω is a canonical formula associated with ω ($Mod(\varphi_\omega) = \{\omega\}$), then $\Box$ and $\Diamond$ are exactly given by:

$$Mod(\Box\varphi) = \{\omega \in \Omega \mid \forall\omega', R(\omega', \omega) \Rightarrow \omega' \models \varphi\}, \tag{14.90}$$

$$Mod(\Diamond\varphi) = \{\omega \in \Omega \mid \exists\omega', R(\omega, \omega'), \omega' \models \varphi\}. \tag{14.91}$$

These equations are similar to the ones used for defining modal operators from an accessibility relation and a structuring element, except that here we consider $R(\omega, \omega')$ for one operator, and $R(\omega', \omega)$ for the other. If R is symmetrical, both are equivalent. In cases where the structuring element (and the accessibility relation) is not symmetrical, we consider its symmetrical in one of the operations.

Modal operators from morphological opening and closing. We can define modal operators from opening and closing on formulas as:

$$\Box\varphi \equiv O(\varphi), \tag{14.92}$$

$$\Diamond\varphi \equiv C(\varphi). \tag{14.93}$$

Unfortunately, this leads to weaker properties than operators derived from erosion and dilation. This comes partly from the fact that no accessibility relation can be derived from opening and closing as easily as from erosion and dilation.

However, it would be interesting to link this approach with the topological interpretation of modal logic as proposed in Aiello and van Benthem, 1999, since opening and closing are related to the notions of topological interior and closure. Note that considering erosion and dilation only leads to a pre-topology (where closure is not idempotent).

Another interesting direction could be to consider the neighbourhood semantics (Aiello and van Benthem, 1999), where here the neighbourhoods of ω would be all elements of the set $N(\omega) = \{B(\omega') \mid \omega' \in \Omega \text{ and } \omega \in B(\omega')\}$. With this semantics, we can prove:

$$\omega \models \Box\varphi \Leftrightarrow \exists\omega' \in \Omega \mid B(\omega') \in N(\omega) \text{ and } B(\omega') \models \varphi. \tag{14.94}$$

The proof of this expression comes from the following rewriting of opening:

$$Mod(\Box\varphi) = \{\omega \in \Omega \mid \exists \omega' \in \Omega,\ \omega \in B(\omega') \text{ and } B(\omega') \models \varphi\}. \tag{14.95}$$

Kripke's semantics can be seen as a particular case, where the neighbourhood of ω is reduced to the singleton $\{B(\omega)\}$.

LEMMA 14.41 *The modal logic constructed from opening and closing satisfies* **T**, **Df**, **D**, **4**, **4c**, **5c**, **N**, **M**, **M'**, **RM**, **RE**, *but not* **5**, **B**, **K**, **C**, **R**, **RR**.

The fact that **K** is not satisfied goes with the interpretation in terms of neighbourhood semantics, which leads to a weaker logic, where **RM** (monotonicity) is satisfied, but not **K** in general (Aiello and van Benthem, 1999).

Extension to the fuzzy case. We now consider fuzzy formulas, i.e. formulas φ for which $Mod(\varphi)$ is a fuzzy subset of Ω and use the fuzzy morphological operators of Bloch and Maître, 1995, (see Sec. 3.4). However, what follows applies as well if other definitions are used.

Modal operators in the fuzzy case can then be constructed from fuzzy erosion and dilation in a similar way as in the crisp case using Eqs. (14.77,14.78). The fuzzy structuring element can be interpreted as a fuzzy relation between worlds. The properties of this fuzzy modal logic are the same as in the crisp case, since fuzzy dilations and erosions have the same properties as the binary ones.

This extension can also be considered from the algebraic point of view of adjunction, based on the results of Deng and Heijmans, 2002, and on a definition of fuzzy erosion in terms of residual implication.

The use of fuzzy structuring elements will appear as particularly useful for expressing intrinsically vague spatial relationships such as directional relative position.

It is also interesting to relate this approach to the possibilistic logic proposed for belief fusion in Boldrin and Saffiotti, 1995, and to similarity-based reasoning (Esteva et al., 1997; Dubois et al., 1997).

4.3 Qualitative representation of spatial relationships and reasoning

For qualitative spatial reasoning, worlds (or interpretations) can represent spatial entities, like regions of the space. Formulas then represent combinations of such entities, and define regions, objects, etc., which may be not connected. For instance, if a formula φ is a symbolic representation of a region X of the space, it can be interpreted for instance as "the object we are looking at is in X". In an epistemic interpretation, it could represent the belief of an agent that the object is in X. The interest of such representations could be also to deal in a qualitative way with any kind of spatial entities, without referring to points.

Using these interpretations, if φ represents some knowledge or belief about a region X of the space, then $\Box\varphi$ represents a restriction of X. If we are looking at an object in X, then $\Box\varphi$ is a necessary region for this object. Similarly, $\Diamond\varphi$ represents an extension of X, and a possible region for the object. In an epistemic interpretation, $\Box\varphi$ can represent the belief of an agent that the object is necessarily in the erosion of X while $\Diamond\varphi$ is the belief that it is possibly in the dilation of X. Interpretations in terms of rough regions are also possible.

In this subsection, we address the problem of qualitative representation of spatial relationships between regions or objects represented by logical formulas.

Topological relationships. Let us first consider topological relationships. Let φ and ψ be two formulas representing two regions X and Y of the space. Note that all what follows holds in both crisp and fuzzy cases. Simple topological relations such as inclusion, exclusion, intersection do not call for more operators than the standard ones of propositional logic (e.g. Bennett, 1995). But other relations such that X *is a tangential part of* Y can benefit from the morphological modal operators. Such a relationship can be expressed as:

$$\varphi \rightarrow \psi \text{ and } \Diamond\varphi \wedge \neg\psi \text{ consistent}, \tag{14.96}$$

or, equivalently,

$$\varphi \rightarrow \psi \text{ and } \varphi \wedge \neg\Box\psi \text{ consistent}. \tag{14.97}$$

Indeed, if X is a tangential part of Y, it is included in Y but its dilation is not, and equivalently it is not included in the erosion of Y, as illustrated in Fig. 14.13.

In a similar way, a relation such that X is a non tangential part of Y is expressed, for a reflexive accessibility relation, as:

$$\Diamond\varphi \rightarrow \psi, \tag{14.98}$$

or, equivalently,

$$\varphi \rightarrow \Box\psi, \tag{14.99}$$

(i.e. in order to verify that X is a non tangential part of Y, we have to prove these relations).

If we also want X to be a proper part, we have to add the following condition:

$$\neg\varphi \wedge \psi \text{ consistent}. \tag{14.100}$$

Let us now consider adjacency (or external connection). Saying that X is adjacent to Y means that they do not intersect and as soon as one region is dilated, it has a non empty intersection with the other one. In symbolic terms, this relation can be expressed as:

(14.101) $\varphi \wedge \psi$ inconsistent and $\Diamond\varphi \wedge \psi$ consistent (or $\varphi \wedge \Diamond\psi$ consistent).

Actually, this expression holds in a discrete domain. If φ and ψ represent spatial entities in a continuous spatial domain, some problems may occur if these entities are closed sets and have parts of local dimension less than the dimension of the space (see Bloch et al., 1997 for a complete discussion). Such problems can be avoided if the entities are reduced to regular ones, i.e. that are equal to the closure of their interior (and by considering an asymptotic definition of adjacency). Using the topological interpretation of modal logic, this amounts to deal with formulas for which we can prove $\varphi \leftrightarrow \Diamond\Box\varphi$.

It is interesting to link these types of representations with the ones developed in the community of mereology and mereotopology, where such relations are defined respectively from parthood and connection predicates (Asher and Vieu, 1995; Randell et al., 1992; Cohn et al., 1997; Varzi, 1996; Renz and Nebel, 2001). Interestingly enough, erosion is defined from inclusion (i.e. a parthood relationship) and dilation from intersection (i.e. a connection relationship). Some axioms of these domains could be expressed in terms of dilation. For instance from a parthood postulate $P(X,Y)$ between two spatial entities X and Y and from dilation δ, tangential proper part could be defined as $TPP(X,Y) = P(X,Y) \wedge \neg P(Y,X) \wedge \neg P(\delta(X),Y)$. Further links certainly deserve to be investigated, in particular with the work presented e.g. in Cohn et al., 1997, Cristani et al., 2000 and Galton, 2000.

Distances. Distances between objects X and Y can be expressed in different forms, as *the distance between X and Y is equal to n*, *the distance between X and Y is less (respectively greater) than n*, *the distance between X and Y is between n_1 and n_2*. Several distances can be related to morphological dilation, as minimum distance and Hausdorff distance, as explained in Sec. 3.3.

Based on algebraic expressions of distances using dilation, the translation into a logical formalism is quite straightforward. Expressing that $d_N(X,Y) = n$ leads to:

(14.102) $$\begin{cases} \forall m < n, \Diamond^m\varphi \wedge \psi \text{ inconsistent, and } \Diamond^m\psi \wedge \varphi \text{ inconsistent} \\ \text{and } \Diamond^n\varphi \wedge \psi \text{ consistent (or } \Diamond^n\psi \wedge \varphi \text{ consistent).} \end{cases}$$

Expressions like $d_N(X,Y) \leq n$ translate into:

(14.103) $\Diamond^n\varphi \wedge \psi$ consistent (or $\Diamond^n\psi \wedge \varphi$ consistent).

Expressions like $d_N(X,Y) \geq n$ translate into:

(14.104) $\forall m < n, \Diamond^m\varphi \wedge \psi$ inconsistent (or $\Diamond^m\psi \wedge \varphi$ inconsistent).

Expressions like $n_1 \leq d_N(X,Y) \leq n_2$ translate into:

$$(14.105) \quad \begin{cases} \forall m < n_1, \Diamond^m \varphi \wedge \psi \text{ inconsistent (or } \Diamond^m \psi \wedge \varphi \text{ inconsistent)} \\ \text{and } \Diamond^{n_2} \varphi \wedge \psi \text{ consistent (or } \Diamond^{n_2} \psi \wedge \varphi \text{ consistent).} \end{cases}$$

Similarly for Hausdorff distance, we translate $H_d(X,Y) = n$ by:

$$(14.106) \quad \begin{cases} \forall m < n, \psi \wedge \neg\Diamond^m \varphi \text{ consistent or } \varphi \wedge \neg\Diamond^m \psi \text{ consistent} \\ \text{and } \psi \rightarrow \Diamond^n \varphi \text{ and } \varphi \rightarrow \Diamond^n \psi. \end{cases}$$

The first condition corresponds to $H_d(X,Y) \geq n$ and the second one to $H_d(X,Y) \leq n$.

Let us consider an example of possible use of these representations for spatial reasoning. If we are looking at an object represented by ψ in an area which is at a distance in an interval $[n_1, n_2]$ of a region represented by φ, this corresponds to a minimum distance greater than n_1 and to a Hausdorff distance less than n_2. This is illustrated in Fig. 14.12.

Then we have to check the following relation:

$$(14.107) \qquad \psi \rightarrow \neg\Diamond^{n_1} \varphi \wedge \Diamond^{n_2} \varphi,$$

or equivalently:

$$(14.108) \qquad \psi \rightarrow \Box^{n_1} \neg\varphi \wedge \Diamond^{n_2} \varphi.$$

This expresses in a symbolic way an imprecise knowledge about distances represented as an interval. If we consider a fuzzy interval, this extends directly by means of fuzzy dilation (see Bloch, 2000c, for detailed expressions of these dilations).

These expressions show how we can convert distance information, which is usually defined in an analytical way, into algebraic expressions through mathematical morphology, and then into logical expressions through morphological expressions of modal operators.

Directional relative position. We use for this relation the same approach as in Sec. 3.4, based on dilation. Let us denote by D^d the dilation corresponding to a directional information in the direction d, and by $\Diamond^d$ the associated modal operator (this assumes that directions can be defined over the set of spatial entities represented as logical formulas). Expressing that an object represented by ψ has to be in direction d with respect to a region represented by φ amounts to check the following relation:

$$(14.109) \qquad \psi \rightarrow \Diamond^d \varphi.$$

In the fuzzy case, this relation can hold to some degree.

Usually for spatial reasoning several relationships have to be used together. This aspect can benefit from the developments in information fusion, both in a numerical and in a logical setting.

Logical reasoning and inference. One of the advantages of logical representations is their inference and reasoning power. Rule-based systems can make use of the proposed representations in a quite straightforward way. But it is also interesting to note that several spatial logics contain ingredients that can be expressed equivalently in morphological terms. We show here some of these links but do not pretend to be exhaustive.

Some links with mereotopology and region connection calculus (RCC) have already been mentioned above. They allow us to combine the expressiveness power of mathematical morphology and the reasoning power of RCC and mereotopology.

The "egg-yolk" structures, as developed e.g. in Cristani et al., 2000 can also lead to interpretations in terms of mathematical morphology. For instance in this model, establishing if a yolk can be a mobile part (in translation) of its egg is based on the notion of congruence. This characterization can be expressed in a very simple way using morphological opening (erosion followed by a dilation): the opening of the egg by the yolk considered as the structuring element should be connected.

Let us now consider two examples of logics of distances. The first one defines a modality $A^{\leq a}$ by Kutz et al., 2002:

$$\omega \models A^{\leq a}\varphi \text{ iff } \forall u, d(\omega, u) \leq a \Rightarrow u \models \varphi, \tag{14.110}$$

where d is a distance between worlds. It is straightforward to show that $A^{\leq a}\varphi$ is equivalent to the erosion of φ by a ball of the distance d of radius a. The dual of $A^{\leq a}$ is equivalent to a dilation. Then we have direct correspondences between the axioms of this distance logics and the axioms of the modal morpho-logics as presented in Bloch, 2002. Some theorems can be also directly deduced from properties of dilation or erosion. For instance, the following is proved to be a theorem:

$$A^{\leq b}\varphi \rightarrow A^{\leq a}\varphi \;\text{ for } a \leq b. \tag{14.111}$$

Using the morphological equivalence, this theorem is directly deduced from the decreasingness of erosion with respect to the size of the structuring element.

The second example concerns nearness logics (Aiello and van Benthem, 2002), where the notion of "nearest than" is modeled as:

$$x \models < N > \varphi, \psi \text{ iff } \exists y, z, (y \models \varphi \wedge z \models \psi) \wedge N(x, y, z) \tag{14.112}$$

where $N(x, y, z)$ means that y is closer to x than z is. The meaning of this expression is that the nearest point distance of x to φ is less than the nearest point distance of x to ψ. An equivalent expression is therefore:

$$x \models \delta_n(\psi) \rightarrow x \models \delta_n(\varphi) \tag{14.113}$$

which expresses that x is reached faster from φ than from ψ by dilations of these formulas.

Other links between linear logics or arrow logics and mathematical morphology exist, as already established in Aiello and van Benthem, 2002.

Finally, let us consider logics of convexity (Aiello and van Benthem, 2002):

$$x \models C\varphi \text{ iff } \exists y, z, (y \models \varphi \wedge z \models \varphi) \wedge (x \in y - z) \tag{14.114}$$

which expresses a linear closure, the iteration of which provides convexity. This iterative closure is clearly equivalent to morphologic closing, where structuring elements are segments in all directions of infinite length (in practice, larger than the largest diameter of the considered spatial entities).

All these examples show interesting links between different spatial logics which have not been exhibited before for most of them. They can be exploited in two ways: the properties of morphological operators can provide additional theorems to these logics; conversely spatial logics endow mathematical morphology with powerful inference and reasoning tools.

Other links between mathematical morphology and non-classical logics are explored in Fujio and Bloch, 2004.

5. Conclusion

This chapter provides an overview of the algebraic basis of mathematical morphology. It shows how this lattice-theoretical formalism can be applied in different frameworks for spatial reasoning, thanks to the representation of shapes and of spatial relations it provides. In particular, it is highly adapted to the modelling of logical relations.

Further links with other chapters in this book are worth to be mentioned, such as mereotopology, modal logics of space and topology.

It should be stressed that we have not given here a complete overview of the methods, techniques and tools of mathematical morphology. Nor have we given any idea of the way to use them (alone or in conjunction with other approaches) in practical image processing problems. This by far exceeds the scope of this chapter. For this, we advise reading classical books such as Heijmans, 1994, Serra, 1982, Serra, 1988 and Soille, 2003, as well as image processing journals and conference proceedings.

Complexity issues have not been addressed in this chapter, except a few words at the end of Sec. 1. The interested reader should consult standard books on morphological image processing (for example, Soille, 2003) for more details on algorithms and data structures for morphology.

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Chapter 15

SPATIAL REASONING AND ONTOLOGY: PARTS, WHOLES, AND LOCATIONS

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Spatial reasoning is no abstract business. It is, to a great extent, reasoning about entities *located in* space, and such entities have spatial *structure*. If the table is in the kitchen, then it follows that the table top is in the kitchen, and it follows because the top is *part of* the table. If the concert took place at the stadium, then it didn't take place in the theater, for concerts are spatially *continuous*. Even when we reason about empty places, we typically do so with an eye to the anatomy of their potential tenants: space as such is perceptually remote and we can hardly understand its structure without imagining what *could* fill the void.

This general feature of our spatial competence might suggest a deep metaphysical truth, to the effect that concrete entities such as objects and events are fundamentally prior to, and independent of, their spatial receptacles. It might even suggest that space itself is just a fiction, a picture of some kind: really there are only objects and events spatially related to one another in various ways. Such was, for instance, the gist of Leibniz's stern *relationism* against Newton's *substantivalism*, in spite of the major role the idea of space plays in the sciences. At the same time, one might argue that our understanding of the spatial structure of objects and events, including their spatial relationhips, depends significantly on our understanding of the structure of space *per se*: that the spatial features we attribute to objects and events are somehow inherited from those of the spatial regions they occupy. Thus, for example, we are inclined to say that ordinary objects have parts insofar as their spatial regions have