Morphological Links Between Formal Concepts and Hypergraphs

Isabelle Bloch(✉)

LTCI, Télécom ParisTech, Université Paris-Saclay, Paris, France
isabelle.bloch@telecom-paristech.fr

Abstract. Hypergraphs can be built from a formal context, and conversely formal contexts can be derived from a hypergraph. Establishing such links allows exploiting morphological operators developed in one framework to derive new operators in the other one. As an example, the combination of derivation operators on formal concepts leads to closing operators on hypergraphs which are not the composition of dilations and erosions. Several other examples are investigated in this paper, with the aim of processing formal contexts and hypergraphs, and navigating in such structures.

Keywords: Formal concept analysis · Hypergraphs · Mathematical morphology operators

1 Introduction

Mathematical morphology on structured representations of information is an active field of research. Given a structured representation, often represented using a graphical model, the classical way to proceed in the deterministic case is to define a partial ordering inducing a lattice structure on this representation, from which adjunctions and algebraic operators are defined. Operators using structuring elements are defined from relationships or distances on the representation[1]. Since each representation has its own semantics and point of view on the information, different definitions of morphological operators were proposed. In this paper, our aim is to establish relationships between previous works on two types of representations: formal concept analysis on the one hand [1,2,5], and hypergraphs on the other hand [6,7]. The idea is to derive a formal context from a hypergraph and conversely, so as to make each formalism inherit from definitions proposed in the other one. A few examples in each direction will be provided. Note that previous work on simplicial complexes [11,12] could be used, by considering simplicial complexes as particular cases of hypergraphs, but this may not be sufficient for our purpose since in general a concept lattice cannot be fully reconstructed from a simplicial complex, as proved in [14].

[1] In this paper we consider only the four basic operators (dilation, erosion, opening, closing).

J. Angulo et al. (Eds.): ISMM 2017, LNCS 10225, pp. 16–27, 2017.
DOI: 10.1007/978-3-319-57240-6_2

In Sect. 2, preliminaries on formal concept analysis and hypergraphs are recalled. A first direction is considered in Sect. 3, with the construction of hypergraphs from a formal context and the derivation of mathematical morphology operators. In Sect. 4, the reverse direction is considered.

2 Preliminaries

In this section, we recall the main definitions and properties of formal concept analysis and hypergraphs, that will be used in this paper.

2.1 Formal Concept Analysis (FCA) [15]

A *formal context* is a triplet $\mathbb{K} = (G, M, I)$, where G is the set of *objects*, M the set of *attributes* or *properties*, and $I \subseteq G \times M$ a *relation* between objects and attributes ($(g, m) \in I$ means that the object g has the attribute m). A *formal concept* of the context $\mathbb{K}$ is a pair (X, Y), with $X \subseteq G$ and $Y \subseteq M$, such that (X, Y) is maximal with the property $X \times Y \subseteq I$. The set X is called the *extent* and the set Y is called the *intent* of the formal concept (X, Y). For any formal concept a, we denote its extent by $e(a)$ and its intent by $i(a)$, i.e. $a = (e(a), i(a))$.

The set of all formal concepts of a given context can be hierarchically ordered by inclusion of their extent (or equivalently by inclusion of their intent):

$$(X_1, Y_1) \preceq (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow Y_2 \subseteq Y_1).$$

This order induces a complete lattice which is called the *concept lattice* of the context (G, M, I), denoted $\mathbb{C}(\mathbb{K})$, or simply $\mathbb{C}$. Infimum and supremum of a family of formal concepts $(X_t, Y_t)_{t \in T}$ are given by:

$$\bigwedge_{t \in T} (X_t, Y_t) = \left(\bigcap_{t \in T} X_t, \alpha(\beta(\bigcup_{t \in T} Y_t)) \right), \tag{1}$$

$$\bigvee_{t \in T} (X_t, Y_t) = \left(\beta(\alpha(\bigcup_{t \in T} X_t)), \bigcap_{t \in T} Y_t \right). \tag{2}$$

For $X \subseteq G$ and $Y \subseteq M$, the *derivation operators* α and β are defined as:

$$\begin{aligned} \alpha(X) &= \{m \in M \mid \forall g \in X, (g, m) \in I\}, \\ \beta(Y) &= \{g \in G \mid \forall m \in Y, (g, m) \in I\}. \end{aligned}$$

The pair (α, β) is a Galois connection between the partially ordered power sets $(\mathcal{P}(G), \subseteq)$ and $(\mathcal{P}(M), \subseteq)$ i.e.

$$\forall X \in \mathcal{P}(G), \forall Y \in \mathcal{P}(M), Y \subseteq \alpha(X) \Leftrightarrow X \subseteq \beta(Y).$$

Saying that (X, Y), with $X \subseteq G$ and $Y \subseteq M$, is a formal concept is equivalent to $\alpha(X) = Y$ and $\beta(Y) = X$.

As a running example, we consider in this paper a set of objects which are integers between 1 and 10, and some of their properties, as displayed in Fig. 1. The table defining I and the corresponding lattice are shown. In this example, the pair $(\{1, 9\}, \{o, s\})$ is a formal concept.

isabelle.bloch@telecom-paristech.fr

$\mathbb{K}$	composite	even	odd	prime	square
1			×		×
2		×		×	
3			×	×	
4	×	×			×
5			×	×	
6	×	×			
7			×	×	
8	×	×			
9	×		×		×
10	×	×			

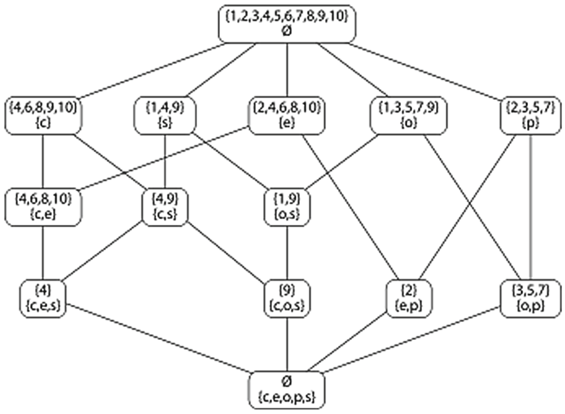

Fig. 1. A simple example of a context and its concept lattice from Wikipedia. Objects are integers from 1 to 10, and attributes are composite (c) (i.e. non prime integer strictly greater than 1), even (e), odd (o), prime (p) and square (s).

2.2 Hypergraphs [3,9]

A *hypergraph* H, denoted by $H = (V, E)$, is defined by a finite set of *vertices* V and a finite family (which can be a multi-set) E of subsets of V called *hyperedges*. The set of vertices forming a hyperedge e, $e \in E$, is denoted by $v(e)$. It is usual to identify a hyperedge and the corresponding set of vertices. If $\cup_{e\in E} v(e) = V$, the hypergraph is without isolated vertex (a vertex x is isolated if $x \in V \setminus \cup_{e\in E} v(e)$). The set of isolated vertices is denoted by $V_{\setminus E}$. By definition the empty hypergraph is the hypergraph $H_\emptyset$ such that $V = \emptyset$ and $E = \emptyset$.

The incidence graph of a hypergraph $H = (V, E)$ is a bipartite graph $IG(H)$ with a vertex set $S = V \sqcup E$ (where $\sqcup$ stands for the disjoint union), and where $x \in V$ and $e \in E$ are adjacent if and only if $x \in v(e)$. Conversely, to each bipartite graph $\Gamma = (V_1 \sqcup V_2, A)$, we can associate two hypergraphs: a hypergraph $H = (V, E)$, where $V = V_1$ and $E = V_2$ and its dual $H^* = (V^*, E^*)$ by exchanging the roles of vertices and hyperedges, where $V^* = V_2$ and $E^* = V_1$.

3 From Formal Contexts to Hypergraphs

In this section we propose a few ways to build hypergraphs from formal contexts. Morphological operators defined on formal contexts then induce operations on hypergraphs.

3.1 Construction of Hypergraphs from a Formal Context

With any context (G, M, I), we can associate a bipartite graph from the disjoint union of objects and properties, and edges defined by the relation I, i.e. $(G \sqcup M, I)$ [4,16], where $\sqcup$ denotes the disjoint union (an extension to the fuzzy case was proposed in [17]). Two vertices $g \in G$ and $m \in M$ are linked if and only if $(g, m) \in I$. This bipartite graph can be considered as the incidence graph of two dual hypergraphs.

isabelle.bloch@telecom-paristech.fr

Definition 1. *Let $\mathbb{K} = (G, M, I)$ be a formal context. Two hypergraphs are defined from $\mathbb{K}$ as:*

1. *$H_1 = (V_1, E_1)$ where the set of vertices V_1 is equal to G (i.e. the objects), and a hyperedge $e \in E_1$ links all objects sharing a given property $m \in M$, i.e. $v(e) = \beta(\{m\})$, where $v(e)$ denotes the set of vertices of e;*
2. *$H_2 = (V_2 = M, E_2 = \{\alpha(\{g\}), g \in G\})$, i.e. the vertices are now properties, and each hyperedge $e \in E_2$ corresponds to an object g and $v(e) = \alpha(\{g\})$.*

Let us consider the example in Fig. 1, and the two aforementioned hypergraphs associated with this context. We have then $V_1 = \{1, \ldots, 10\}$, and for instance $s \in E_1$ and $v(s) = \{1, 4, 9\}$. Similarly $V_2 = \{c, e, o, p, s\}$. A hyperedge corresponding to object 5 is the subset of vertices $v(5) = \{o, p\}$. Note that in this example we have multiple hyperedges. In particular we also have $v(7) = \{o, p\}$, since objects 5 and 7 have the same set of properties (the two corresponding lines in the table in Fig. 1 are the same). If hypergraphs without repeated hyperedges are considered as preferable, they can be obtained by making the context non redundant, by clarification (removing in particular identical lines and columns in the table for this example).

Instead of considering the bipartite graph defined from the relation I, hyperedges can be built on $G \sqcup M$ from the formal concepts, which provides another interesting hypergraph.

Definition 2. *Let $\mathbb{C}$ be the concept lattice associated with the formal context $\mathbb{K} = (G, M, I)$. We define a hypergraph associated with $\mathbb{C}$ as $H = (V = G \sqcup M, E = \mathbb{C})$, i.e. a hyperedge is formed by the subsets X and Y of G and M respectively, such that $(X, Y) \in \mathbb{C}$ (X and Y are linked if $\alpha(X) = Y$ and $\beta(Y) = X$). The set of vertices of a hyperedge e is then denoted by $v(e) = \{g \in X\} \sqcup \{m \in Y\}$.*

Graphically, the hyperedges of this hypergraph H correspond to the elements of the lattice, as displayed for the number example in Fig. 1. For instance $\{1, 9, o, s\}$ is a hyperedge of H.

3.2 Morphological Operators

As shown in [1,5] (and previously mentioned in [8]), there are some links between derivation operators and Galois connections on the one hand, and morphological operators and adjunctions on the other hand. This was extended to the fuzzy case in [2]. In particular, the derivation operators α and β are anti-dilations, and the compositions $\alpha\beta$ and $\beta\alpha$ are closings. The Galois connection property between α and β corresponds to the adjunction property between a dilation and an erosion, by reversing the ordering on one of the two spaces. These links are summarized in Table 1[2]. In this section, we further explore how operations on formal concepts, defined from $\mathcal{P}(G)$ into $\mathcal{P}(M)$, from $\mathcal{P}(M)$ into $\mathcal{P}(G)$, or directly on $\mathbb{C}$, lead to operations on hypergraphs.

[2] In the table we denote by $Inv(\varphi)$ the set of fixed points of an operator φ (i.e. $x \in Inv(\varphi)$ iff $\varphi(x) = x$).

Table 1. Similarities between some mathematical morphology notions and formal concept analysis [1].

Adjunctions, dilations and erosions	Galois connection, derivation operators
$\delta\colon (\mathcal{L}, \preceq) \to (\mathcal{L}', \preceq')$, $\varepsilon\colon (\mathcal{L}', \preceq') \to (\mathcal{L}, \preceq)$	$\alpha\colon \mathcal{P}(G) \to \mathcal{P}(M)$, $\beta\colon \mathcal{P}(M) \to \mathcal{P}(G)$
$\delta(x) \preceq' y \iff x \preceq \varepsilon(y)$	$X \subseteq \beta(Y) \iff Y \subseteq \alpha(X)$
Increasing operators	Decreasing operators
$\varepsilon\delta\varepsilon = \varepsilon$, $\delta\varepsilon\delta = \delta$	$\alpha\beta\alpha = \alpha$, $\beta\alpha\beta = \beta$
$\varepsilon\delta =$ closing (closure operator), $\delta\varepsilon =$ opening (kernel operator)	$\alpha\beta$ and $\beta\alpha =$ both closure operators (closings)
$Inv(\varepsilon\delta) = \varepsilon(\mathcal{L}')$, $Inv(\delta\varepsilon) = \delta(\mathcal{L})$	$Inv(\alpha\beta) = \alpha\big(\mathcal{P}(G)\big)$, $Inv(\beta\alpha) = \beta\big(\mathcal{P}(M)\big)$
$\varepsilon(\mathcal{L}')$ is a Moore family, $\delta(\mathcal{L})$ is a dual Moore family	$\alpha\big(\mathcal{P}(G)\big)$ and $\beta\big(\mathcal{P}(M)\big)$ are Moore families (or closure systems)
δ is a dilation: $\delta(\vee x_i) = \vee'\big(\delta(x_i)\big)$	α is an anti-dilation: $\alpha(\cup X_i) = \cap\alpha(X_i)$
ε is an erosion: $\varepsilon(\wedge' y_i) = \wedge\big(\varepsilon(y_i)\big)$	β is an anti-dilation: $\beta(\cup Y_i) = \cap\beta(Y_i)$

Derivation Operators, Dilations and Anti-dilations. Let us first interpret the derivation operators in terms of morphological operators on hypergraphs. Considering H_1, for a singleton $g \in G$, we have $\alpha(\{g\}) = \{m \in M \mid g \in v(m)\}$, in this hypergraph. This means that with each g we associate the set of hyperedges which contain g.

Proposition 1. *The derivation operator α applied on singletons is equivalent to the dilation on hypergraphs introduced in Example 4 of [6], defined from $(\mathcal{P}(V_1), \subseteq)$ into $(\mathcal{P}(E_1), \subseteq)$ as:*

$$\forall g \in V_1, \delta(\{g\}) = \{e \in E_1 \mid g \in v(e)\} \text{ and } \forall X \subseteq V_1, \delta(X) = \bigcup_{g \in X} \delta(\{g\}).$$

We have $\alpha(\{g\}) = \delta(\{g\})$, and for any subset X of G, $\bigcup_{g \in X} \alpha(\{g\}) = \delta(X)$.

As a consequence, since $\alpha(X) = \bigcap_{g \in X} \alpha(\{g\})$, α is the anti-dilation, counterpart of this dilation δ on hypergraphs.

A similar reasoning applies for $Y \in M$, by considering now dilations from $(\mathcal{P}(E_1), \subseteq)$ into $(\mathcal{P}(V_1), \subseteq)$ and corresponding links with β.

By exchanging the roles of G and M, the above interpretations of derivation operators on H_1 are transposed on H_2.

Summarizing, α is interpreted as an anti-dilation from $\mathcal{P}(V_1)$ into $\mathcal{P}(E_1)$ (or equivalently from $\mathcal{P}(E_2)$ into $\mathcal{P}(V_2)$), and β as an anti-dilation from $\mathcal{P}(E_1)$ into $\mathcal{P}(V_1)$ (or equivalently from $\mathcal{P}(V_2)$ into $\mathcal{P}(E_2)$), in the sense of hypergraphs.

The combinations $\beta\alpha$ and $\alpha\beta$ are closings on $\mathcal{P}(G)$ or $\mathcal{P}(M)$, which directly define closings on the derived hypergraphs H_1 and H_2. These are new definitions of closings, enriching the ones previously proposed in [6,7].

To illustrate the effect of $\beta\alpha$, let us consider the example in Fig. 2, where the represented hypergraph is derived from a formal context, as H_1, i.e. the vertices correspond to objects and the hyperedges to properties. In this figure, vertices are represented by dots, and hyperedges by closed lines including the vertices composing them. Let us consider the vertex x_1 (red dot in the figure). We have $\alpha(\{x_1\}) = \delta(\{x_1\}) = \{m \in E_1 \mid (x_1, m) \in I\} = \{e_1, e_5\}$, and $\beta\alpha(\{x_1\}) = \beta(\{e_1, e_5\}) = \{g \in V_1 \mid (g, e_1) \in I \text{ and } (g, e_5) \in I\} = v(e_1) \cap v(e_5) = \{x_1, x_2\}$ (i.e. the red and magenta vertices in the figure). As a comparison, if we consider the adjoint erosion of δ to perform the closing, we have $\varepsilon\delta(\{x_1\}) = \bigcup\{X \in \mathcal{P}(V_1) \mid \forall x \in X, \delta(\{x\}) \subseteq \delta(\{x_1\})\} = v(e_1) \cup v(e_5)$. This shows that the two closings may provide completely different results.

Similarly, let us consider the following example for $\alpha\beta$: $\beta(\{e_1\}) = \{x \in V_1 \mid x \in v(e_1)\} = v(e_1)$, i.e. all colored points (red, magenta and cyan in Fig. 2), and $\alpha\beta(\{e_1\}) = \{m \in E_1 \mid \forall x \in \beta(\{e_1\}), x \in v(m)\} = \{e_1\}$. Considering adjoint erosion and dilation, different results could be obtained.

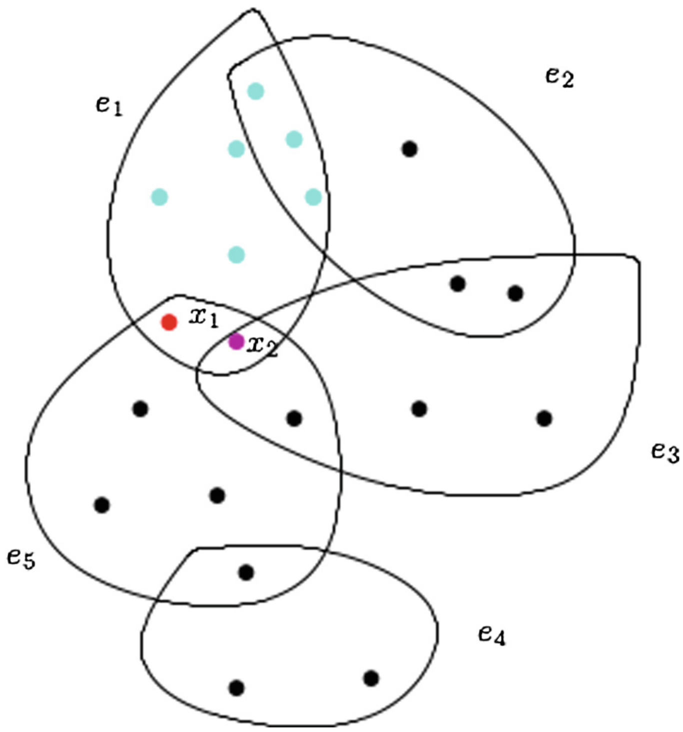

Fig. 2. Example of hypergraph and closing defined from derivation operators: $\alpha(\{x_1\}) = \{e_1, e_5\}$ and $\beta\alpha(\{x_1\}) = \{x_1, x_2\}$; $\alpha\beta(\{e_1\}) = \{e_1\}$. (Color figure online)

Interpretation of Morphological Operations on Formal Concepts in Terms of Hypergraphs. In [1,2], morphological operators were introduced, based on structuring elements defined either from I or from a distance. Let us take as a structuring element centered at $m \in M$, or a neighborhood of m, the set of $g \in G$ such that $(g, m) \in I$ (and conversely the set of $m \in M$ such that $(g, m) \in I$ is a neighborhood of g). Operators δ_I and ε_I^* from $\mathcal{P}(M)$ into $\mathcal{P}(G)$, and δ_I^* and ε_I from $\mathcal{P}(G)$ into $\mathcal{P}(M)$ were defined as:

$$\forall X \in \mathcal{P}(G), \forall Y \in \mathcal{P}(M)$$

$$\begin{aligned}
\delta_I(Y) &= \{g \in G \mid \exists m \in Y, (g,m) \in I\}, \\
\varepsilon_I(X) &= \{m \in M \mid \forall g \in G, (g,m) \in I \Rightarrow g \in X\}, \\
\delta_I^*(X) &= \{m \in M \mid \exists g \in X, (g,m) \in I\}, \\
\varepsilon_I^*(Y) &= \{g \in G \mid \forall m \in M, (g,m) \in I \Rightarrow m \in Y\}.
\end{aligned}$$

The pairs of operators $(\varepsilon_I, \delta_I)$ and $(\varepsilon_I^*, \delta_I^*)$ are adjunctions (and δ_I and δ_I^* are dilations, ε_I and ε_I^* are erosions). These operators also correspond to possibilistic interpretations of formal concepts, as proposed in [13]. Moreover, the following duality relations hold: $\delta_I(M \setminus Y) = G \setminus \varepsilon_I^*(Y)$ and $\delta_I^*(G \setminus X) = M \setminus \varepsilon_I(X)$.

Interpreting now G and M as sets of vertices and hyperedges of a hypergraph, we come up with morphological operations on hypergraphs, from either the set of vertices to the set of hyperedges or the converse, as also developed in [10] for graphs, and [6,7] for hypergraphs. Let us consider H_1, where vertices correspond to objects and hyperedges to properties. We have the following interpretations:

- $\forall Y \in \mathcal{P}(E_1), \delta_I(Y) = \bigcup_{m \in Y} \beta(\{m\}) = \bigcup_{m \in Y} v(m)$, i.e. all vertices defining the hyperedges in Y. This corresponds to Example 3 in [6].
- $\forall X \in \mathcal{P}(V_1), \varepsilon_I(X) = \{m \in E_1 \mid v(m) \subseteq X\}$, which is the adjoint erosion ε' of the dilation δ' defined as $\forall Y \in \mathcal{P}(E_1), \delta'(Y) = \bigcup_{e \in Y} v(e)$ (see Proposition 11 in [7]). The result is the set of complete hyperedges (i.e. with all vertices contained in the hyperedges) formed by vertices of V_1. The corresponding opening γ' is the set of vertices of these hyperedges. Coming back to the formal concepts, this means that in a given subset X of objects, we remove by this opening the objects which are in incomplete hyperedges, i.e. which have properties shared by objects which are not in X.
- $\forall X \in \mathcal{P}(V_1), \delta_I^*(X) = \bigcup_{g \in X} \alpha(\{g\}) = \bigcup_{g \in X} \{m \in E_1 \mid g \in v(m)\}$. It corresponds to the dilation introduced in Example 3 in [6]. See also Proposition 1.
- $\forall Y \in \mathcal{P}(E_1), \varepsilon_I^*(Y) = \{g \in V_1 \mid \alpha(\{g\}) \subseteq Y\}$, which is the set of vertices such that all hyperedges including these vertices are in Y.

Similar interpretations hold for the second construction $H_2 = (V_2, E_2)$.

Morphological Operations on Hypergraph Representations of Formal Concepts. Morphological operations from distances and neighborhoods, as defined on formal contexts in [1], can be proposed considering the representation as a hypergraph of formal concepts $H = (V = G \sqcup M, E = \mathbb{C})$, as described above.

Let us consider operators defined from a distance on $\mathbb{C}$. Several distances were introduced in [1,2], for instance from valuations ω_G and ω_M defined as the cardinality of the extent ($\omega_G(a) = |e(a)|$) or the intent ($\omega_M(a) = |i(a)|$) of a formal concept a:

$$\begin{aligned}
&\forall (a_1, a_2) \in \mathbb{C}^2, d_{\omega_G}(a_1, a_2) = 2\omega_G(a_1 \wedge a_2) - \omega_G(a_1) - \omega_G(a_2), \\
&\forall (a_1, a_2) \in \mathbb{C}^2, d_{\omega_M}(a_1, a_2) = \omega_M(a_1) + \omega_M(a_2) - 2\omega_M(a_1 \vee a_2),
\end{aligned}$$

where $\wedge$ and $\vee$ are the infimum and supremum of formal concepts (Eqs. 1 and 2). These two functions are metrics on $\mathbb{C}$ (this also holds in the fuzzy case [2]). Structuring elements defined as balls of these distances can then be used as structuring elements. An example of dilation is illustrated in Fig. 3 (from [2]), using either d_{ω_G} or d_{ω_M}.

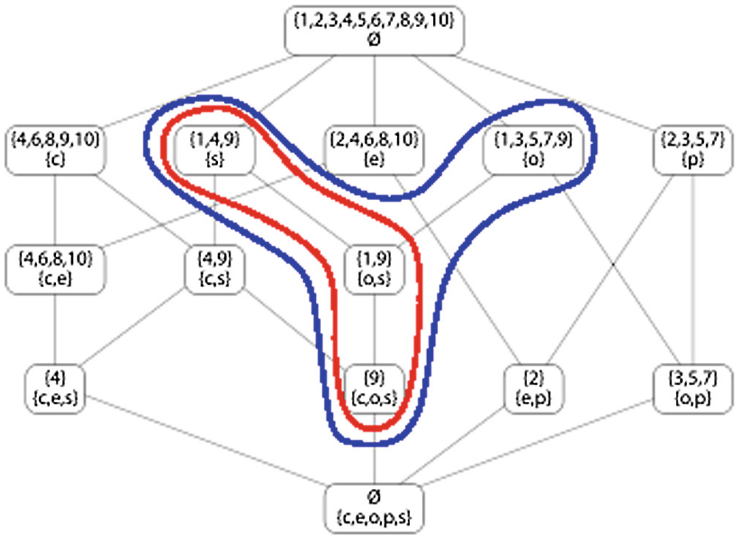

Fig. 3. Dilation of $\{a\} = \{(\{1,9\},\{o,s\})\}$ using a ball of d_{ω_G} (red) and of d_{ω_M} (blue) as structuring element [2]. (Color figure online)

As mentioned above, each element of $\mathbb{C}$ can be interpreted as a hyperedge of a hypergraph $H = (V = G \sqcup M, E = \mathbb{C})$. Hence these dilations induce dilations from $\mathcal{P}(E)$ on $\mathcal{P}(E)$, which are again new operators.

Another definition in [2] relies on the decomposition of each formal concept as the disjunction of join-irreducible elements. Dilating each of these irreducible elements (for instance using balls of d_{ω_G} or d_{ω_M} as structuring elements), defines new operations, which in turn induce new dilations on hypergraphs. An example is reproduced from [2] in Fig. 4, with a direct interpretation in terms of hypergraphs (i.e. dilation from $\mathcal{P}(E)$ into $\mathcal{P}(E)$). In this example, the concept $a_1 = (\{1,4,9\},\{s\})$ is decomposed into irreducible elements as $a_1 = (\{4\},\{c,e,s\}) \vee (\{1,9\},\{o,s\}) \vee (\{9\},\{c,o,s\})$ and each element of the decomposition is dilated using an elementary ball of d_{ω_G} as structuring element.

Another example of dilation from $\mathcal{P}(E)$ into $\mathcal{P}(E)$ can be defined as follows:

$$\forall e \in E, \delta(\{e\}) = \{e' \in E \mid v(e) \sqcap v(e') \neq \emptyset\}, \tag{3}$$

and $\forall A \subseteq E, \delta(A) = \bigcup_{e \in A} \delta(\{e\})$, with the pseudo non empty intersection defined as:

$$v(X,Y) \sqcap v(X',Y') \neq \emptyset \Leftrightarrow X \cap X' \neq \emptyset \text{ and } Y \cap Y' \neq \emptyset.$$

The conjunction of the two constraints allows limiting the neighborhood of a concept and hence the extent of the dilation. For this example, a hypergraph

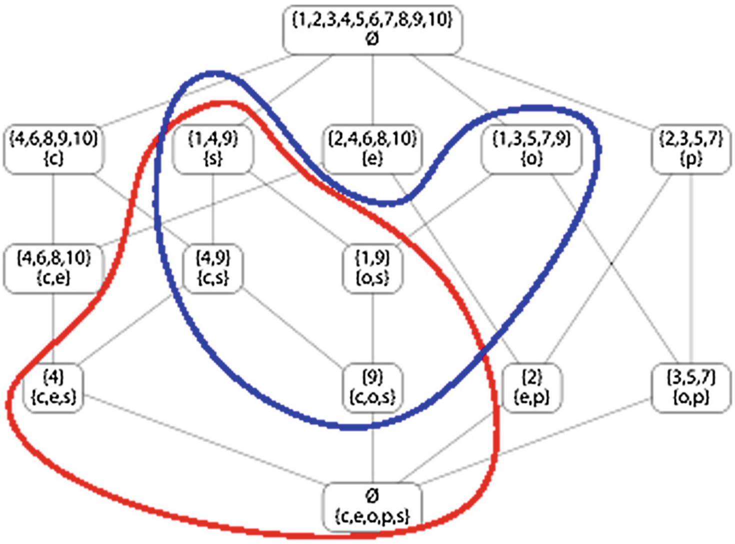

Fig. 4. Dilation of $\{a_1\} = \{(\{1, 4, 9\}, \{s\})\}$ using a ball of d_{ω_G} as structuring element for each irreducible element of its decomposition [2] (red), and dilation of $(\{1, 9\}, \{0, s\})$ using Eq. 3 (blue). (Color figure online)

is first built from a formal context, then morphological operators are defined on the hypergraph, which induce operators on the original concept lattice. This approach can therefore be considered as being in-between the ones in this section and in the next one.

In the example in Fig. 1, the dilation of the formal concept $(\{1, 9\}, \{0, s\})$ is:

$$\delta(\{1, 9\}, \{0, s\}) = \{(X', Y') \in \mathbb{C} \mid X' \cap \{1, 9\} \neq \emptyset \text{ and } Y' \cap \{o, s\} \neq \emptyset\} =$$
$$\{(\{1, 4, 9\}, \{s\}), (\{1, 3, 5, 7, 9\}, \{o\}), (\{4, 9\}, \{c, s\}), (\{1, 9\}, \{o, s\}), (\{9\}, \{c, o, s\})\}.$$

Note that this dilation, illustrated in Fig. 4 (in blue) is different from the one that can be built on the graph defined by the concept lattice (as depicted in Fig. 1), where the dilation of a concept would include all the concepts linked directly by an edge in the graph (then $(\{4, 9\}, \{c, s\})$ would not be included in the dilation of $(\{1, 9\}, \{0, s\})$. It is also different from the dilations illustrated in Fig. 3.

We can also limit the extent of dilations by limiting the number of changes to go from one concept to another one.

4 From Hypergraphs to Formal Contexts

4.1 Construction of Formal Contexts from a Hypergraph

Conversely, formal contexts can be defined from a hypergraph $H = (V, E)$.

Definition 3. *Let $H = (V, E)$ be a hypergraph. Two formal contexts are defined from H, by setting*

1. *either $G = V$, $M = E$, and $\forall g \in G, \forall m \in M, (g, m) \in I$ iff $g \in v(m)$,*
2. *or $G = E$, $M = V$, and $\forall g \in G, \forall m \in M, (g, m) \in I$ iff $m \in v(g)$.*

isabelle.bloch@telecom-paristech.fr

The relation I corresponds to the incidence matrix of H, and the derivation operators α and β can be expressed equivalently using I or using $v(m)$ (or $v(e)$). For instance in the first construction:

$$\forall X \subseteq G = V, \alpha(X) = \{m \in M = E \mid \forall g \in X, g \in v(m)\}; \tag{4}$$

$$\forall Y \subseteq M = E, \beta(Y) = \{g \in G = V \mid \forall m \in Y, g \in v(m)\}. \tag{5}$$

Then, as for any formal context, formal concepts are defined as pairs $(A, B), A \subseteq G = V, B \subseteq M = E$ such that $\alpha(A) = B$ and $\beta(B) = A$. Similar expressions hold for the second construction.

4.2 Morphological Operators

Several morphological operators on hypergraphs have been proposed in [6,7]. Thanks to the two constructions above, they yield directly operators on the derived formal contexts.

Let us consider the operators introduced in the examples of [6], and re-interpret them in terms of formal concepts. In Example 1, the structuring element is defined as the set of hyperedges intersecting the considered one, and we have for each m in M:

$$\delta(\{m\}) = \{m' \in M \mid v(m) \cap v(m') \neq \emptyset\} = \{m' \in M \mid \beta(\{m\}) \cap \beta(\{m'\}) \neq \emptyset\},$$

which represents all properties that have at least one object in common with m. This definition is similar to the one used in Eq. 3. The dilation of any subset of M is defined as the disjunction of the dilations of its elements. The adjoint erosion is given by:

$$\forall Y \in \mathcal{P}(M), \varepsilon(Y) = \bigcup\{Y' \mid \forall m \in Y', \delta(\{m\}) \subseteq Y\}.$$

Note that these operators are defined from $\mathcal{P}(M)$ into $\mathcal{P}(M)$. Operators from $\mathcal{P}(V)$ into $\mathcal{P}(V)$ can be defined in a similar way.

Let us consider the property "composite" in the number example (Fig. 1). We have $\delta(\{c\}) = \{c, e, s, o\}$, $\varepsilon(\{c, e, s, 0\}) = \{c, s\}$ (the dilation of all singletons is provided in Table 2), i.e. the morphological closing of $\{c\}$ is $\varepsilon\delta(\{c\}) = \{c, s\}$.

Table 2. Elementary dilations of the properties in the example of Fig. 1.

m	c	e	o	p	s
$\delta(\{m\})$	$\{c, e, s, o\}$	$\{e, p, c, s\}$	$\{o, s, p, c, s\}$	$\{p, e, o\}$	$\{s, o, c, e\}$

As mentioned in Example 2 in [6], a constraint on the cardinality of the intersection between $v(m)$ and $v(m')$ can be added to limit the extent of the dilation:

$$\delta_k(\{m\}) = \{m' \in M \mid |v(m) \cap v(m')| \geq k\} = \{m' \in M \mid |\beta(\{m\}) \cap \beta(\{m'\})| \geq k\}.$$

This means that we consider in the dilation all properties that have at least k objects in common with m. Such operations can be useful for clustering applications, among others.

We have for instance, for $k = 2$ and $k = 3$, $\delta_2(\{c\}) = \{c, e, s\}, \delta_3(\{c\}) = \{c, e\}$.

As in [6], we can also define similar operations from $\mathcal{P}(M)$ into $\mathcal{P}(V)$ by considering as result of the dilation all vertices forming the hyperedges in the above definitions. The compositions with the adjoint erosion define closings on $\mathcal{P}(M)$ or on $\mathcal{P}(V)$ that are different from $\alpha\beta$ and $\beta\alpha$.

As another example, let us consider the opening γ in Proposition 10 of [7] on the power set of vertices as:

$$\forall X \in \mathcal{P}(V), \gamma(X) = \bigcup \{B_e \mid e \in E, B_e \subseteq X\}, \tag{6}$$

where $\forall e \in E, B_e = \cup\{v(e') \mid e' \in E, v(e') \cap v(e) \neq \emptyset\}$. This opening acts as a filter that removes at least vertices that belong to incomplete hyperedges in X (i.e. for which the set of vertices is not completely included in X). Interpreting the hypergraph as a formal concept leads to a natural interpretation in terms of object filtering: the opening γ applied on a subset of G removes objects that have a property shared by other objects, among which at least one is not in the considered subset. Complete hyperedges can also be removed for this opening (in contrary to γ' above).

5 Outlook

In this paper, we highlighted some straightforward links between formal concepts and hypergraphs. These two ways of representing structured information correspond to different points of view (starting with the graphical representations), that suggest different ways of defining morphological operators. A few examples were given, showing that the links between the two frameworks allow each one to benefit from the other. In particular, operations that may seem very natural in one setting provide new operators in the other one, such enriching the toolbox for manipulating formal concepts and hypergraphs.

This paper presents a preliminary work, that can be developed in several directions, to further explore new morphological operators on the one hand, in particular based on different types of distances, and to derive useful applications on the other hand. Examples of applications include filtering, redundancy elimination, clustering, operations robust to small changes and associated similarities, etc.

Acknowledgments. This work has been partly supported by the French ANR LOGIMA project.

References

1. Atif, J., Bloch, I., Distel, F., Hudelot, C.: Mathematical morphology operators over concept lattices. In: Cellier, P., Distel, F., Ganter, B. (eds.) ICFCA 2013. LNCS (LNAI), vol. 7880, pp. 28–43. Springer, Heidelberg (2013). doi:10.1007/978-3-642-38317-5_2
2. Atif, J., Bloch, I., Hudelot, C.: Some relationships between fuzzy sets, mathematical morphology, rough sets, F-transforms, and formal concept analysis. Int. J. Uncertain. Fuzz. Knowl. Based Syst. **24**(2), 1–32 (2016)
3. Berge, C.: Hypergraphs. Elsevier Science Publisher, Amsterdam (1989)
4. Berry, A., Sigayret, A.: Representing a concept lattice by a graph. Discret. Appl. Math. **144**(1), 27–42 (2004)
5. Bloch, I.: Mathematical morphology, lattices, and formal concept analysis. In: 8th International Conference on Concept Lattices and Their Applications (CLA 2011) - Invited Conference, p. 1, Nancy, France, October 2011
6. Bloch, I., Bretto, A.: Mathematical morphology on hypergraphs, application to similarity and positive kernel. Comput. Vis. Image Underst. **117**(4), 342–354 (2013)
7. Bloch, I., Bretto, A., Leborgne, A.: Robust similarity between hypergraphs based on valuations and mathematical morphology operators. Discret. Appl. Math. **183**, 2–19 (2015)
8. Bloch, I., Heijmans, H., Ronse, C.: Mathematical morphology. In: Aiello, M., Pratt-Hartman, I., van Benthem, J. (eds.) Handbook of Spatial Logics, pp. 857–947. Springer, Dordrecht (2007). Chap. 13
9. Bretto, A.: Hypergraph Theory: An Introduction. Mathematical Engineering. Springer, Cham (2013)
10. Cousty, J., Najman, L., Dias, F., Serra, J.: Morphological filtering on graphs. Comput. Vis. Image Underst. **117**, 370–385 (2013)
11. Dias, F., Cousty, J., Najman, L.: Some morphological operators on simplicial complex spaces. In: Debled-Rennesson, I., Domenjoud, E., Kerautret, B., Even, P. (eds.) DGCI 2011. LNCS, vol. 6607, pp. 441–452. Springer, Heidelberg (2011). doi:10.1007/978-3-642-19867-0_37
12. Dias, F., Cousty, J., Najman, L.: Dimensional operators for mathematical morphology on simplicial complexes. Pattern Recogn. Lett. **47**, 111–119 (2014)
13. Dubois, D., Dupin de Saint-Cyr, F., Prade, H.: A possibility-theoretic view of formal concept analysis. Fundamenta Informaticae **75**(1), 195–213 (2007)
14. Freund, A., Andreatta, M., Giavitto, J.L.: Lattice-based and topological representations of binary relations with an application to music. Ann. Math. Artif. Intell. **73**(3–4), 311–334 (2015)
15. Ganter, B., Wille, R., Franzke, C.: Formal Concept Analysis: Mathematical Foundations. Springer-Verlag New York, Inc., New York (1997)
16. Gaume, B., Navarro, E., Prade, H.: A parallel between extended formal concept analysis and bipartite graphs analysis. In: Hüllermeier, E., Kruse, R., Hoffmann, F. (eds.) IPMU 2010. LNCS (LNAI), vol. 6178, pp. 270–280. Springer, Heidelberg (2010). doi:10.1007/978-3-642-14049-5_28
17. Ghosh, P., Kundu, K., Sarkar, D.: Fuzzy graph representation of a fuzzy concept lattice. Fuzzy Sets Syst. **161**(12), 1669–1675 (2010)