CONSTRUCTING A FUZZY MATHEMATICAL MORPHOLOGY : ALTERNATIVE WAYS

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***Abstract* : We propose in this paper two methods for constructing a mathematical morphology working on fuzzy sets with fuzzy structuring elements. They are compatible with classical mathematical morphology on binary sets or grey-level functions with binary structuring elements. Their properties are presented and comparisons are made between the two approaches. Fuzzy mathematical morphology provides new operations on fuzzy sets and can be applied in particular for introducing spatial uncertain information in a decision process for data fusion.**

INTRODUCTION

In image analysis and pattern recognition, fuzzy sets proved to be useful representations for segmenting, classifying or data fusing, whenever regions or classes cannot be crisply defined [3]. Since their introduction, they have been widely used for control and decision making and more recently for images. In order to manipulate fuzzy sets as extensions of crisp sets, operations have been developed which generalize set operations (inclusion, intersection, union, etc.) [9]. Then, the need for geometrical operations on fuzzy sets appeared and concepts like convexity, perimeter, area, distance have been developed. They are summarized in [3].

In this paper, we are interested in the extension of mathematical morphology for fuzzy sets. Mathematical morphology originally aims at studying shapes by comparing them locally with structuring elements through set relations. So, for dealing with fuzzy sets, it naturally follows set and geometrical operations. Fuzzy sets can be treated as functions through their membership function. However, a direct application of erosion and dilation for functions [4] is not satisfactory as the result of such an operation is no longer the membership function of a fuzzy set.

Mathematical morphology for fuzzy sets has already been addressed in [3] where shrinking and expanding of fuzzy sets are defined but only with binary structuring elements. They exactly correspond to grey-level erosion and dilation with a binary structuring element applied to the membership function of a fuzzy set. Fuzzy structuring elements have not been considered.

A completely different approach has been described in [8]. It relies on the association of classical mathematical morphology with order statistics. It does not provide exactly fuzzy erosion and dilation but a series of transformations ranging continuously from grey-level erosion by the support of the structuring element to grey-level dilation by the same structuring element.

Erosion and dilation of fuzzy sets by fuzzy structuring elements are defined in [5] and [6]. They are strongly related to the classical erosion and dilation on functions, using a similar definition but guaranteeing the internal property when applied to fuzzy sets. Therefore, they exhibit similar properties than conventional morphology for functions. These definitions have been derived keeping in mind the point of view of mathematical morphology mostly. Thus, their advantages lie mainly in their ability to generate other operations like opening and closing which are idempotent, increasing and anti-extensive (respectively extensive) and in their contribution to theoretical mathematical morphology (algebraic properties, representation theorems) [7]. When looking at the contribution of these definitions from a fuzzy set or a decision theoretic point of view, their implications may appear somewhat weak or not in agreement with the idea we intuitively have of uncertainty propagation (see part 3), and alternative definitions may be looked for, which emphasize more the fuzziness properties.

The aim of this paper is to present these other definitions for fuzzy dilation and erosion, which were developed in order to provide new transformations on fuzzy sets which include spatial information reasoning within the scope of decision making. These definitions are deduced from two different principles of construction. The first definition (section 1) relies on the concept of fuzzification, which represents a fuzzy set membership function as an integral of characteristic functions of crisp sets. The second one (section 2) is deduced from a functional expression of set relations (a similar approach is used in [5]). Section 3 is dedicated to the comparison between the two definitions and also with Sinha and Dougherty definition. Only the main results will be given here, complete proofs can be found in [1].

I. FUZZY EROSION AND DILATION BY FUZZIFICATION

A. Principle

Let M be the set of all fuzzy set defined on a space E, or equivalently the set of all their membership functions μ from E to $[0,1]$. Let M_C be the set of all binary (or crisp sets), or equivalently the set of all their characteristic functions μ_C from E to $[0,1]$.

A cut at level α of a fuzzy set characterized by μ is the binary set with characteristic function μ_α. A fuzzy set can be reconstructed from its own cuts by: $\forall x \in E,\ \mu(x) = \int_0^1 \mu_\alpha(x) d\alpha$. Let Φ be a fuzzy function, from M to M. Φ is the fuzzification of a function Φ_C from M_C to M_C on binary sets if the restriction of Φ to M_C is equal to Φ_C. The fuzzification of a binary function can be obtained in a way analogous to the reconstruction of a fuzzy set from its own cuts by: $\forall \mu \in M,\ \Phi(\mu) = \int_0^1 \Phi_C(\mu_\alpha) d\alpha$.

This fuzzification principle is the basis for constructing our first fuzzy dilation and erosion from the binary definitions.

B. Definitions

Definition 1.a : let μ and ν be membership functions of two fuzzy sets. The dilation of μ by ν is obtained by fuzzification over μ then over ν, or, equivalently, over ν then over μ :

$$d_\nu(\mu)(x) = \int_0^1 d_\nu(\mu_\beta)(x) d\beta = \int_0^1 \int_0^1 d_{\nu_\alpha}(\mu_\beta)(x) d\alpha d\beta$$
$$= \int_0^1 d_{\nu_\alpha}(\mu)(x) d\alpha = \int_0^1 \int_0^1 d_{\nu_\alpha}(\mu_\beta)(x) d\beta d\alpha.$$

A simple straightforward derivation provides: $\int_0^1 d_{\nu_\alpha}(\mu)(x) d\alpha = \int_0^1 \sup_{y \in (\nu_\alpha)_x} \mu(y) d\alpha$ and thus, for any $x \in E$:

$$\boxed{d_\nu(\mu)(x) = \int_0^1 \sup_{y \in (\nu_\alpha)_x} \mu(y) d\alpha} \qquad \text{(def. 1)}$$

The result $d_\nu(\mu)$ is the membership function of a fuzzy set.

In a similar way, erosion can be defined by fuzzification.

Definition 1.b : let μ and ν be two fuzzy sets, the erosion of μ by ν is obtained by a double fuzzification, and we have, for any $x \in E$:

$$\boxed{e_\nu(\mu)(x) = \int_0^1 \inf_{y \in (\nu_\alpha)_x} \mu(y) d\alpha}$$

The result $e_\nu(\mu)$ is the membership function of a fuzzy set.

C. Properties

Intuitively, the global behaviour of the fuzzy dilation is as follows : membership degrees are increased of these points which were not "surely" belonging to the fuzzy set, also local maxima are propagated within a given neighbourhood. Furthermore a regularization effect exists which makes the resulting curves "smoother" than the original ones.

More precisely, following properties hold :

Proposition 1 : let $\bar{\mu}$ denote the symmetrical of μ with respect to the origin of the space E. Using Minkowski addition commutativity, we obtain :

$$d_\nu(\mu) = d_{\bar{\mu}}(\bar{\nu}) = \overline{(d_\mu(\nu))}.$$

This equation expresses a pseudo-commutativity.

Proposition 2 : the fuzzy dilation $d_\nu(\mu)$ is increasing for fuzzy set inclusion with respect to the initial fuzzy set μ and with respect to the structuring element ν (fuzzy inclusion is defined by the less-or-equal relation on membership functions) :

$$\forall (\mu,\mu') \in M^2,\ \mu \le \mu' \Rightarrow \forall \nu \in M,\ d_\nu(\mu) \le d_\nu(\mu'),$$
$$\forall (\nu,\nu') \in M^2,\ \nu \le \nu' \Rightarrow \forall \mu \in M,\ d_\nu(\mu) \le d_{\nu'}(\mu).$$

Proposition 3 : the fuzzy set dilation is extensive if $\nu(0) = 1$, ν being the structuring element :

$$\forall (\nu,\mu) \in M^2,\ \nu(0) = 1 \Rightarrow d_\nu(\mu) \ge \mu.$$

Simple examples can show that dilation is no longer extensive if $\nu(0) \ne 1$.

Proposition 4 : dilation of the intersection of two fuzzy sets :

$$\forall (\mu,\mu',\nu) \in M^3,\ d_\nu(\mu \cap \mu') \le d_\nu(\mu) \cap d_\nu(\mu').$$

Proposition 5 : dilation of the union of two fuzzy sets :

$$\forall (\mu,\mu',\nu) \in M^3,\ d_\nu(\mu \cup \mu') \ge d_\nu(\mu) \cup d_\nu(\mu').$$

Proposition 6 : dilation by the intersection of two fuzzy structuring elements :

$$\forall (\mu,\nu,\nu') \in M^3,\ d_{\nu \cap \nu'}(\mu) \le d_\nu(\mu) \cap d_{\nu'}(\mu).$$

Proposition 7 : dilation by the union of two fuzzy structuring elements :

$$\forall (\mu,\nu,\nu') \in M^3,\ d_{\nu \cup \nu'}(\mu) \ge d_\nu(\mu) \cup d_{\nu'}(\mu).$$

Proposition 8 : regularization effect : if the membership function μ decays in $O(x^n)$ and if ν decays in $O(x^m)$, then $d_\nu(\mu)$ decays in $O(x^{n+m})$, at least for some values of x in E.

Note that if ν is binary, the inequalities in propositions 5 and 7 become equalities.

Similar properties hold for fuzzy erosion :

Proposition 9 : fuzzy erosion $e_\nu(\mu)$ is increasing with respect to the initial fuzzy set μ and decreasing with respect to the structuring element ν.

Proposition 10 : fuzzy erosion is anti-extensive if $\nu(0) = 1$, ν being the structuring element :

$$\forall (\nu,\mu) \in M^2,\ \nu(0) = 1 \Rightarrow e_\nu(\mu) \le \mu.$$

Proposition 11 : erosion of the intersection of two fuzzy sets :

$$\forall (\mu,\mu',\nu) \in M^3,\ e_\nu(\mu \cap \mu') \le e_\nu(\mu) \cap e_\nu(\mu').$$

Proposition 12 : erosion of the union of two fuzzy sets :

$$\forall (\mu,\mu',\nu) \in M^3,\ e_\nu(\mu \cup \mu') \ge e_\nu(\mu) \cup e_\nu(\mu').$$

Proposition 13 : erosion by the intersection of two fuzzy structuring elements :

$$\forall (\mu,\nu,\nu') \in M^3,\ e_{\nu \cap \nu'}(\mu) \ge e_\nu(\mu) \cup e_{\nu'}(\mu).$$

Proposition 14 : erosion by the union of two fuzzy structuring elements :

$$\forall (\mu,\nu,\nu') \in M^3,\ e_{\nu \cup \nu'}(\mu) \le e_\nu(\mu) \cap e_{\nu'}(\mu).$$

As for dilation, if ν is crisp, equalities hold in propositions 11 and 14.

The properties for erosion can be either proved directly or deduced from the properties on dilation by using following important duality result :

Proposition 15 : fuzzy dilation and fuzzy erosion are dual operations with respect to fuzzy set complementation :

$$\forall(\mu,\nu)\in M^2,\ \forall x\in E,\ d_\nu(1-\mu)(x) = 1 - e_\nu(\mu)(x).$$

As stated in [4], mathematical morphology aims at satisfying four basic principles : compatibility with translations, compatibility with homotheties, local knowledge and semi-continuity. These four principles are satisfied by the previous definitions, and we have for example for dilation :

Proposition 16 : for a fixed μ, the transformations which associate to any ν of M the dilation $d_\nu(\mu)$ and the erosion $e_\nu(\mu)$, satisfy to the principle of compatibility with translations. Similarly, for a fixed ν, the transformations which associate to any μ of M the dilation $d_\nu(\mu)$ and the erosion $e_\nu(\mu)$ satisfy to this principle. This means that transformations like erosion or dilation do not depend on the origin of space E where fuzzy sets are defined.

Proposition 17 : for a fixed μ, the transformations which associate to any ν of M the dilation $d_\nu(\mu)$ and the erosion $e_\nu(\mu)$, satisfy to the principle of compatibility with homotheties. Similarly, for a fixed ν, the transformations which associate to any μ of M the dilation $d_\nu(\mu)$ and the erosion $e_\nu(\mu)$ satisfy to this principle.

Compatibility with homotheties guarantees that the transformations do not depend on a scale parameter λ. In the case of fuzzy sets, this scale parameter is limited to $]0,1]$ so that the result remains a fuzzy set: $(\lambda\mu)(x) = \lambda\mu(x)$.

Proposition 18 : the computation of dilation and erosion of a fuzzy set μ in a binary mask Z is sufficient to derive the result of the operation in the erosion of Z by the union of cuts at level α of the structuring element for $\alpha\in]0,1]$ (i.e. its support) :

$$[d_\nu(\mu\cap Z)]\cap(e_{\bigcup_{\alpha\in]0,1]}\nu_\alpha}(Z)) = d_\nu(\mu)\cap(e_{\bigcup_{\alpha\in]0,1]}\nu_\alpha}(Z)),$$

$$[e_\nu(\mu\cap Z)]\cap(e_{\bigcup_{\alpha\in]0,1]}\nu_\alpha}(Z)) = e_\nu(\mu)\cap(e_{\bigcup_{\alpha\in]0,1]}\nu_\alpha}(Z)).$$

Proposition 19 : fuzzy dilation and erosion by a fixed structuring element ν are semi-continuous operations. Let $(\mu_i)_{i\in N}$ be a series of fuzzy sets, decreasing with respect to fuzzy inclusion ($\leq$), with the conditions that: $\lim_{i\to+\infty}\mu_i = \mu$ and $\forall i\in N,\ \mu_i\leq\mu$. Then, the series $(d_\nu(\mu_i))_{i\in N}$ is decreasing and $\lim_{i\to+\infty} d_\nu(\mu_i) = d_\nu(\mu)$.

This principle ensures the robustness of transformations.

II. Fuzzy Erosion and Dilation by Functional Union and Intersection

A. Principle

A second definition of fuzzy dilation can be obtained from the definition of binary dilation of X by a structuring element B expressed by $\{x\in E, B_x\cap X\neq\varnothing\}$ by translating it in functional terms using the characteristic functions μ_X and μ_B of X and B. In this way, we obtain :

$$B_x\cap X\neq\varnothing \Leftrightarrow \exists y\in B_x,\ y\in X \Leftrightarrow \sup_{y\in E}\min[\mu_B(y-x),\mu_X(y)] = 1.$$

Similarly for erosion, the translation of the set equality $B_x\subset X$ in a functional expression provides :

$$B_x\subset X \Leftrightarrow \inf_{y\in E}\max[1-\mu_B(y-x),\mu_X(y)] = 1.$$

B. Definitions

Definition 2.a : from the above principle, the dilation of a fuzzy set μ by a fuzzy structuring element ν (denoted $d^2{}_\nu(\mu)$) is defined, for any $x\in E$, by :

$$\boxed{d^2{}_\nu(\mu)(x) = \sup_{y\in E}\min[\mu(y),\nu(y-x)]} \quad \text{(def. 2)}$$

This definition is by construction compatible with the binary dilation. If ν is binary and for any μ, we have $d^2{}_\nu(\mu)(x) = \sup_{y\in\nu_x}\mu(y)$, and thus the definition is also compatible with grey-level dilation (with binary structuring element).

Definition 2.b : in the same way, fuzzy erosion can be defined by :

$$\boxed{e^2{}_\nu(\mu)(x) = \inf_{y\in E}\max[\mu(y),1-\nu(y-x)]}$$

As for dilation, compatibility with binary erosion and grey-level erosion with binary structuring element holds.

C. Properties

Proposition 20 : fuzzy dilation and fuzzy erosion are dual operations with respect to fuzzy set complementation :

$$\forall(\mu,\nu)\in M^2,\ \forall x\in E,\ d^2{}_\nu(1-\mu)(x) = 1 - e^2{}_\nu(\mu)(x).$$

This proposition allows to deduce the properties of fuzzy erosion from the properties of fuzzy dilation, which are the only ones given here.

Proposition 21 : the fuzzy dilatation is increasing with respect to μ and to ν.

Proposition 22 : the fuzzy dilation is extensive if $\nu(0) = 1$.

Proposition 23 : the fuzzy dilation is pseudo-commutative, in the sense of proposition 1.

Proposition 24 : dilation of the union of two fuzzy sets :

$$\forall(\mu,\mu',\nu)\in M^3,\ d^2{}_\nu(\mu\cup\mu') = d^2{}_\nu(\mu)\cup d^2{}_\nu(\mu').$$

Proposition 25 : dilation of the intersection of two fuzzy sets :

$$\forall(\mu,\mu',\nu)\in M^3,\ d^2{}_\nu(\mu\cap\mu') \leq d^2{}_\nu(\mu)\cap d^2{}_\nu(\mu').$$

Similar results hold for the dilation by the union or intersection of two fuzzy structuring elements.

Proposition 26 : for this definition, we have a very strong result on the cuts of a dilated fuzzy set :

$\forall(\mu,\nu)\in M^2,\ \forall\alpha\in[0,1],\ [d^2{}_\nu(\mu)]_\alpha = d^2{}_{\nu_\alpha}(\mu_\alpha).$

Proposition 27 : the fuzzy dilation satisfies the principle of compatibility with translations.

Proposition 28 : the principle of local knowledge is satisfied :

$$[d^2{}_\nu(\mu \cap Z)] \cap (e^2{}_{\bigcup_{\alpha\in]0,1]} \nu_\alpha}(Z)) = d^2{}_\nu(\mu) \cap (e^2{}_{\bigcup_{\alpha\in]0,1]} \nu_\alpha}(Z)).$$

Proposition 29 : the fuzzy dilation is semi-continuous.

However, this second definition of fuzzy dilation does not satisfy the principle of compatibility with homotheties. But it is not a major drawback from the point of view of fuzzy sets.

III. Comparison between the two Definitions

A. Underlying concepts

The underlying concepts of the two construction principles are fully different. They have in common a basic requirement : we want to construct operations which are compatible with the binary case. This requirement has been achieved in two different ways :

- by considering a fuzzy set as a stack of binary sets ;

- by translating a set equality into a functional one.

We gave one example for each construction principle. Other ones could have been given. For the first one, the formal expression for "stack" was an integral. It could have been a "sup" for example and then, the membership function of a fuzzy set would have taken the form $\mu(x) = sup\{\alpha\in[0,1]/\mu_\alpha(x)=1\}$.

Similarly, the translation of set equalities into functional relations can be performed in several ways. They correspond to different expressions of union and intersection for fuzzy sets. The translation given here corresponds to the most used definitions of union as a "max" and intersection as a "min". Other generalizations may be used, such as product for intersection and algebraic sum for union, or bounded sum for union, each one giving rise to another fuzzy mathematical morphology :

$$\boxed{d^3{}_\nu(\mu)(x) = \sup_{y\in E} [\mu(y)\nu(y-x)]} \qquad \text{(def. 3)}$$

$$\boxed{e^3{}_\nu(\mu)(x) = \inf_{y\in E} [\mu(y)\nu(y-x)+1-\nu(y-x)]}$$

In a similar way, the approach of [5] relies on the union expressed as a bounded sum. The variety of approaches is due to the variety of ways for finding operations on functions which are equivalent to union or intersection if the functions take only values 1 and 0 [2]. Table 1 presents several fuzzy union and intersection (which satisfy the duality principle) and the corresponding fuzzy erosion and dilation. The last line corresponds to the definition of [5].

Figure 1 shows the qualitative differences between the definitions of dilation on a simple example. For a particular application, the definition has to be chosen depending on the desired effects and on the required properties (which are slightly different, see Table 2). In particular the definition of [5] illustrated on Figure 1.c has, sometimes, stronger morphological properties but we see from Figure 1.c that it has weaker effect on uncertain propagation.

B. Comparison of the properties of the two definitions

Table 2 summarizes the properties of the two definitions of the fuzzy dilation. They are also compared to the two other definitions given in Table 1. The right column corresponds to the definition found in [5]. A similar comparison can be made for erosion. Iteration and combination are the basis for deriving further operations from erosion and dilation. For the classical definition, the iteration of two successive dilations is associative. This result does not hold for fuzzy dilation in the general case. In the same way, the succession of an erosion and a dilation is not anti-extensive and thus does not define an algebraic opening. The line "combination" in Table 2 shows if the combination of erosion and dilation provides algebraic opening and closing. However, even if it does not, these operations have interesting effects (of reducing noise for example) which can be appreciated in a context of noisy data fusion.

Generally speaking, it is not surprising that some properties are lost when extending a theory. Table 2 shows however that both definitions behave generally well with respect to the properties of mathematical morphology. Most important is that they provide a sound basis for morphological operations on fuzzy sets, which allows to introduce spatial information in a decision making framework.

Union	Intersection	Erosion	Dilation	Definition
$\max(\mu,\nu)$	$\min(\mu,\nu)$	$\inf_{y\in E} \max[\mu(y), 1-\nu(y-x)]$	$\sup_{y\in E} \min[\mu(y),\nu(y-x)]$	Def. 2
$\mu+\nu-\mu\nu$	$\mu\nu$	$\inf_{y\in E} [\mu(y)\nu(y-x)+1-\nu(y-x)]$	$\sup_{y\in E} [\mu(y)\nu(y-x)]$	Def. 3
$\min(1,\mu+\nu)$	$\max(0,\mu+\nu-1)$	$\inf_{y\in E} \min[1, 1+\mu(y)-\nu(y-x)]$	$\sup_{y\in E} \max[0,\mu(y)+\nu(y-x)-1]$	[5]

Table 1

	$\int_0^1 \sup_{y \in (\nu_\alpha)_x} \mu(y) d\alpha$ (def.1)	$\sup_{y \in E} \min[\mu(y), \nu(y-x)]$ (def.2)	$\sup_{y \in E} [\mu(y)\nu(y-x)]$ (def.3)	$\sup_{y \in E} \max[0, \mu(y)+\nu(y-x)-1]$ ([5])
Compatibility with $\mu \oplus \nu$ if ν is binary	yes	yes	yes	yes
Increasingness	yes	yes	yes	yes
Extensivity	if $\nu(0)=1$	if $\nu(0)=1$	if $\nu(0)=1$	if $\nu(0)=1$
Pseudo-commutativity	yes	yes	yes	yes
Intersection	$\leq$	$\leq$	$\leq$	$\leq$
Union	$\geq$	$=$	$=$	$=$
Cuts	×	$[d^2{}_\nu(\mu)]_\alpha = d^2{}_{\nu_\alpha}(\mu_\alpha)$	×	×
Compatibility with translations	yes	yes	yes	yes
Compatibility with homotheties	yes	no	yes	no
Local knowledge	yes	yes	yes	yes
Semi-continuity	yes	yes	yes	yes
Erosion	$\int_0^1 \inf_{y \in (\nu_\alpha)_x} \mu(y) d\alpha$	$\inf_{y \in E} \max[\mu(y), 1-\nu(y-x)]$	$\inf_{y \in E} [\mu(y)\nu(y-x)+1-\nu(y-x)]$	$\inf_{y \in E} \min[1, 1+\mu(y)-\nu(y-x)]$
Duality	yes	yes	yes	yes
Iteration	no	yes	yes	yes
Combination	no	no	no	yes
Support of dilation	$supp(\mu) \oplus supp(\nu)$	$supp(\mu) \oplus supp(\nu)$	$supp(\mu) \oplus supp(\nu)$	$supp(\mu) \oplus supp(\nu)$
Regularization (if $\mu \propto x^n$ and $\nu \propto x^m$)	$\propto x^{n+m}$	$\propto x^{m\frac{\min(n,m)}{\max(n,m)}}$	$\propto x^{n+m}$	$\propto x^{\max(n,m)}$
Complexity	$O(NV)$	$O(NA)$	$O(NA)$	$O(NA)$

Table 2

The complexity is given in the finite case, where N is the cardinality of E and A the cardinality of the support of the structuring element $supp(\nu)$.

IV. Conclusion

We proposed in this paper two definitions for fuzzy mathematical morphology. They extend the set of possible operations on fuzzy sets by adding morphological ones, taking into account a fuzzy neighbourhood. Erosion and dilation have been shown to have good properties with respect to fuzzy sets and to mathematical morphology for the both approaches presented in sections 1 and 2. Only a few properties usually required in mathematical morphology are weakened or lost. These new definitions may be used as alternatives to either binary mathematical morphology as in [3], or grey-tone mathematical morphology adapted to fuzzy sets as in [5]. They provide operations which have been tested on multisource medical image data fusion, and exhibit excellent properties, well in agreement with the intuitive notion of spatial uncertainty management which is one of the components of decision making in pattern recognition.

V. References

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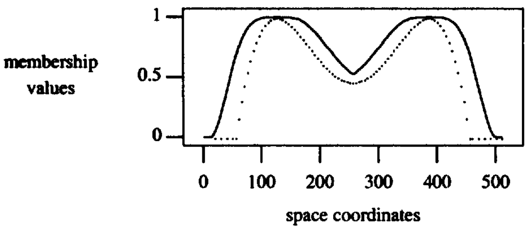


Figure 1.a : $\int_0^1 \sup_{y \in (\nu_\alpha)_x} \mu(y) d\alpha$ (def. 1).

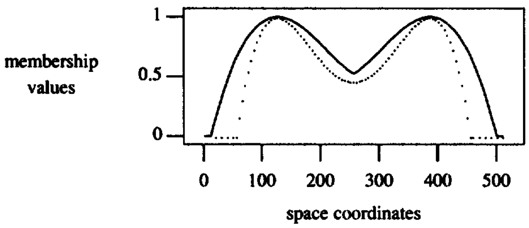


Figure 1.b : $\sup_{y \in E} \min[\mu(y), \nu(y-x)]$ (def. 2).

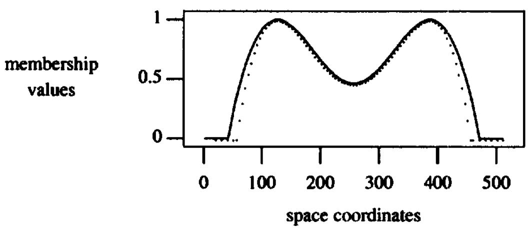


Figure 1.c : $\sup_{y \in E} \max[0, \mu(y)+\nu(y-x)-1]$ (def. of [5]).

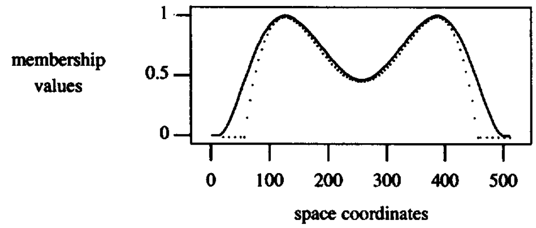
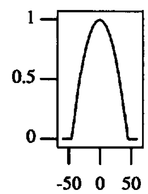


Figure 1.d : $\sup_{y \in E} [\mu(y)\nu(y-x)]$ (def. 3).

(The initial fuzzy set is dashed and the result of dilation is drawn with solid line).



Structuring element used for Figure 1.