Duality vs Adjunction and General Form for Fuzzy Mathematical Morphology

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Abstract. We establish in this paper the link between the two main approaches for fuzzy mathematical morphology, based on duality with respect to complementation and on the adjunction property, respectively. We also prove that the corresponding definitions of fuzzy dilation and erosion are the most general ones if a set of classical properties is required.

1 Introduction

Extending mathematical morphology to fuzzy sets was addressed by several authors during the last years. Some definitions just consider grey levels as membership functions, or use binary structuring elements. Here we restrict ourselves to really fuzzy approaches, where fuzzy sets have to be transformed according to fuzzy structuring elements. Initial developments can be found in the definition of fuzzy Minkowski addition [1]. Then this problem has been addressed by several authors independently, e.g. [2,3,4,5,6,7,8,9]. These works can be divided into two main approaches. In the first one [2], an important property that is put to the fore is the duality between erosion and dilation. A second type of approach is based on the notions of adjunction and fuzzy implication, and was formalized in [8]. The aim of this paper is twofold. First, we will clarify the links between both approaches (which are summarized in Section 2) and establish the conditions of their equivalence (Section 3). Then, in Section 4, we will prove that the definitions of dilation and erosion in these approaches are the most general ones if we want them to share a set of classical properties with standard mathematical morphology.

2 Summary of the Two Main Approaches

Let us first briefly recall the two main approaches. Fuzzy sets are defined on a space $\mathcal{S}$, through their membership functions from $\mathcal{S}$ into $[0, 1]$. The set of fuzzy sets on $\mathcal{S}$ is denoted by $\mathcal{F}$, and $\leq$ is the partial ordering defined by $\mu \leq \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) \leq \nu(x)$. This defines a lattice $(\mathcal{F}, \leq)$.

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2.1 Fuzzy Morphology by Formal Translation Based on t-Norms and t-Conorms

The first attempts to build fuzzy mathematical morphology were based on translating binary equations into fuzzy ones, as developed in [2]. This translation is done term by term, by substituting all crisp expressions by their fuzzy equivalents. For instance, intersection is replaced by a t-norm, union by a t-conorm, sets by fuzzy set membership functions, etc.

An important property that was put to the fore in this approach is the duality between erosion and dilation. We consider here morphological dilation and erosion, i.e. based on a structuring element.

Let $\varepsilon_B(X)$ denote the erosion of the set X by B, defined by $x \in \varepsilon_B(X) \Leftrightarrow B_x \subseteq X$, where B_x denotes B translated at point x. The translation of this expression into fuzzy terms leads to a natural way to define the erosion of a fuzzy set μ by a fuzzy structuring element ν, as:

$$\forall x \in \mathcal{S},\ \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} T[c(\nu(y-x)), \mu(y)], \tag{1}$$

were T is a t-conorm and c a complementation. This corresponds to a degree of inclusion of ν, translated at x, in μ. The dual of erosion in the crisp case is $\delta_B(X) = (\varepsilon_{\check{B}}(X^c))^c$, where $\check{B}$ denotes the symmetrical of B with respect to the origin. Accordingly, by duality with respect to the complementation c, fuzzy dilation is then defined by:

$$\forall x \in \mathcal{S},\ \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} t[\nu(x-y), \mu(y)], \tag{2}$$

where t is the t-norm associated to the t-conorm T with respect to the complementation c. This definition of dilation corresponds to the translation of the following set equivalence: $x \in \delta_B(x) \Leftrightarrow \check{B}_x \cap X \neq \emptyset \Leftrightarrow \exists y \in \mathcal{S},\ y \in \check{B}_x \cap X$. The fuzzy dilation at x is expressed as the degree of intersection of ν translated at x and μ, which is dual of the degree of inclusion used for the erosion. These forms of fuzzy dilation and fuzzy erosion are very general, and several definitions found in the literature appear as particular cases, such as [5,3,10] (see e.g. [2,11] for a comparison).

Finally, fuzzy opening (respectively fuzzy closing) is simply defined as the combination of a fuzzy erosion followed by a fuzzy dilation (respectively a fuzzy dilation followed by a fuzzy erosion), by using dual t-norms and t-conorms.

The detail of properties of these definitions can be found in [2]. Most properties of classical morphology are satisfied whatever the choice of t and T. But in order to get true closing and opening, i.e. which are extensive (respectively anti-extensive) and idempotent, a necessary and sufficient condition on t and T is $t[b, T(c(b), a)] \leq a$, which is satisfied for Lukasiewicz t-norm and t-conorm for instance.

2.2 Fuzzy Morphology Using Adjunction and Residual Implications

A second type of approach is based on the notions of adjunction and fuzzy implication. Here the algebraic framework is the main guideline, which contrasts with the previous approach where duality was imposed in first place.

Fuzzy implication is often defined as [12]: $Imp(a, b) = T[c(a), b)]$. Fuzzy inclusion, as used in the previous approach, and therefore fuzzy erosion, is related to implication by the following equation: $\mathcal{I}(\nu, \mu) = \inf_{x \in \mathcal{S}} Imp[\nu(x), \mu(x)]$.

This suggests another way to define fuzzy erosion (and dilation), by using other forms of fuzzy implication. One interesting approach is to use residual implications: $Imp(a, b) = \sup\{\varepsilon \in [0,1], t(a, \varepsilon) \leq b\}$. This provides the following expression for the degree of inclusion: $\mathcal{I}(\nu, \mu) = \inf_{x \in \mathcal{S}} \sup\{\varepsilon \in [0,1], t(\nu(x), \varepsilon) \leq \mu(x)\}$. This definition coincides with the previous one for particular forms of t, typically Lukasiewicz t-norm.

The derivation of fuzzy morphological operators from residual implication has been proposed in [4], and then developed e.g. in [7]. One of its main advantages is that it leads to idempotent fuzzy closing and opening. This approach was formalized from the algebraic point of view of adjunction in [8]. It has then been used by other authors, e.g. [9]. This leads to general algebraic fuzzy erosion and dilation. Let us detail this approach. A fuzzy implication I is a mapping from $[0,1] \times [0,1]$ into $[0,1]$ which is decreasing in the first argument, increasing in the second one and satisfies $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$. A fuzzy conjunction is a mapping from $[0,1] \times [0,1]$ into $[0,1]$ which is increasing in both arguments and satisfies $C(0,0) = C(1,0) = C(0,1) = 0$ and $C(1,1) = 1$. If C is also associative and commutative, it is a t-norm. A pair of operators (I, C) are said adjoint if:

$$C(a, b) \leq c \Leftrightarrow b \leq I(a, c). \tag{3}$$

The adjoint of a conjunction is a residual implication.

Fuzzy dilation and erosion are then defined as:

$$\forall x \in \mathcal{S},\ \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} C(\nu(x - y), \mu(y)), \tag{4}$$

$$\forall x \in \mathcal{S},\ \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} I(\nu(y - x), \mu(y)). \tag{5}$$

Note that (I, C) is an adjunction if and only if $(\varepsilon_\nu, \delta_\nu)$ is an adjunction on the lattice $(\mathcal{F}, \leq)$ for any ν.

Opening and closing derived from these operations by combination have all required properties, whatever the choice of C and I. Some properties of dilation, such as iterativity, require C to be associative and commutative, i.e. a t-norm. This will be further investigated in Section 4.

3 Links Between Both Approaches

3.1 Dual vs Adjoint Operators

If C is a t-norm, then the dilation in the second approach is exactly the same as the one obtained in the first approach. To understand further the relation between both approaches for erosion, we define

$$\hat{I}(a, b) = I(c(a), b).$$

Then $\hat{I}$ is increasing in both arguments, and if I is further assumed to satisfy $I(a,b) = I(c(b), c(a))$ and $I(c(I(a,b)), d) = I(a, I(c(b), d))$, then $\hat{I}$ is commutative and associative, hence a t-conorm. In the following, in order to simplify notations we simply take $c(a) = 1 - a$ which is the most usual complementation, but the derivations and results hold for any c.

Equation 5 can be rewritten as:

$$\varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} \hat{I}(1 - \nu(y-x), \mu(y)),$$

which corresponds to the fuzzy erosion of the first approach. The adjunction property can also be written as:

$$C(a,b) \leq c \Leftrightarrow b \leq \hat{I}(1-a, c).$$

However, pairs of dual t-norms and t-conorms are not identical to pairs of adjoint operators. Let us take a few examples. For $C = \min$, its adjoint is $I(a,b) = b$ if $b < a$, and 1 otherwise (known as Gödel implication). But the derived $\hat{I}$ is the dual of the conjunction defined as $C(a,b) = 0$ if $b \leq 1-a$ and b otherwise. Conversely, the adjoint of this conjunction is $I(a,b) = \max(1-a, b)$ (Kleene-Dienes implication), the dual of which is the minimum conjunction. Lukasiewicz operators $C(a,b) = \max(0, a+b-1)$ and $\hat{I}(a,b) = \min(1, a+b)$ are both adjoint and dual, which explains the exact correspondence between both approaches for these operators. Table 1 summarizes the differences between dual and adjoint operators.

Table 1. A few dual and adjoint operators: dual and adjoint are generally not identical, except in the case of Lukasiewicz operators (among these examples)

conjunction	dual t-conorm	adjoint implication I	$\hat{I}$
$\min(a,b)$	$\max(a,b)$	$\begin{cases} b \text{ if } b < a \\ 1 \text{ otherwise} \end{cases}$ (Gödel)	$\begin{cases} b \text{ if } b < 1-a \\ 1 \text{ otherwise} \end{cases}$
$\begin{cases} 0 \text{ if } b \leq 1-a \\ b \text{ otherwise} \end{cases}$	$\begin{cases} b \text{ if } b < 1-a \\ 1 \text{ otherwise} \end{cases}$	$\max(1-a, b)$ (Kleene-Dienes)	$\max(a,b)$
$\max(0, a+b-1)$	$\min(1, a+b)$	$\min(1, 1-a+b)$ (Lukasiewicz)	$\min(1, a+b)$

3.2 Equivalence Condition

The first main result of this paper is expressed in the following theorem.

Theorem 1. *The condition for dual t-norms and t-conorms leading to idempotent opening and closing (i.e.* $t(b, T(1-b, a)) \leq a$*) is equivalent to the adjunction property between* C *and* I *for* $t = C$ *and* $T = \hat{I}$.

Proof. Let us assume that the adjunction property is satisfied for $t = C$ and $T = \hat{I}$, i.e.

$$t(a, b) \leq c \Leftrightarrow b \leq T(1 - a, c). \quad (6)$$

Applying this property to the tautology $T(1 - b, a) \leq T(1 - b, a)$ leads directly to:

$$t(b, T(1 - b, a)) \leq a. \quad (7)$$

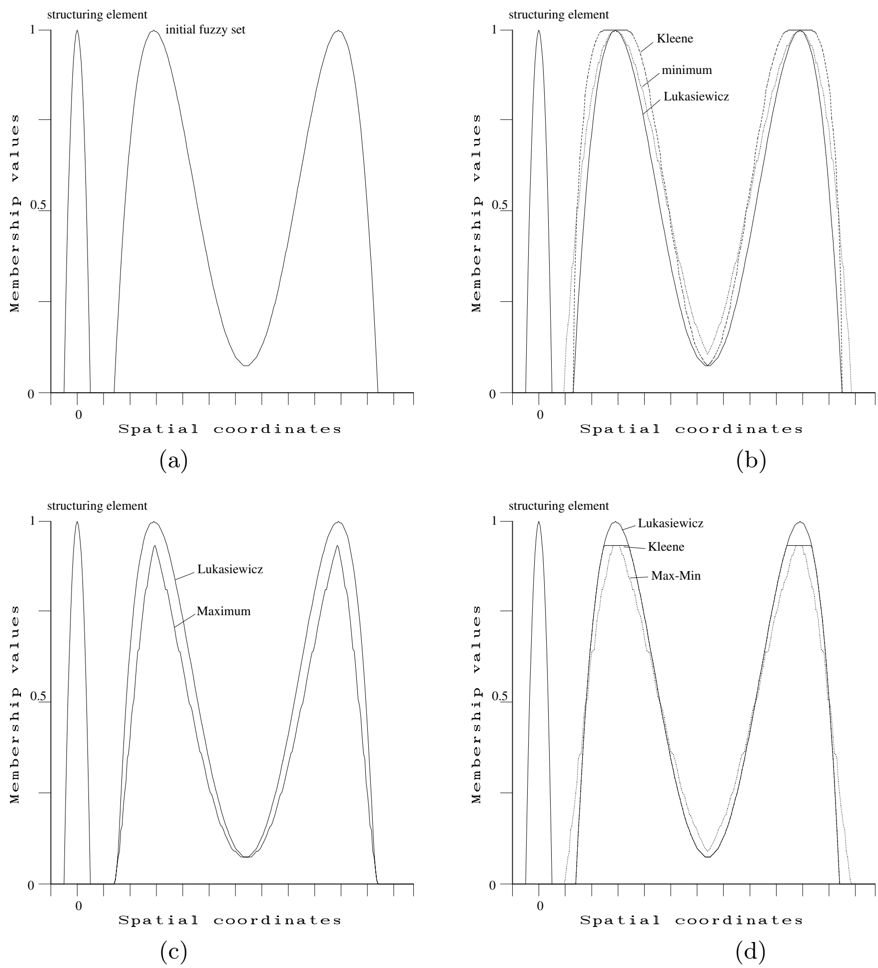

Fig. 1. Illustration of some morphological operations on a one-dimensional example. (a) Initial fuzzy set and fuzzy structuring element. (b) Dilations using the minimum, Lukasiewicz and Kleene-Dienes conjunctions. (c) Erosions using the maximum and Lukasiewicz t-conorms. (d) Opening using max-min, Lukasiewicz and Kleene-Dienes operators.

Let us now assume that we have the property expressed by Equation 7 for dual operators. If $b \leq T(1-a, c)$, then since t is increasing, we have $t(a, b) \leq t(a, T(1-a, c))$ which is less than c by Eq. 7. This implies $t(a, b) \leq c$.

Since t and T are dual, Eq. 7 is equivalent to $1 - T(1-b, 1-T(1-b, a)) \leq a$, and, by exchanging the roles of $1-a$ and a and then of a and b, to $T(1-a, t(a, b)) \geq b$.

Now, if $t(a, b) \leq c$, since T is increasing, we have $T(1-a, t(a, b)) \leq T(1-a, c)$. Since the first term is greater than b, this implies $b \leq T(1-a, c)$. ■

This new result completes the link between both approaches by showing that duality and adjunction are generally not compatible, and that in case dual operators lead to true opening and closing, the condition on these operators is equivalent to the adjunction property. This means that in case duality and adjunction are compatible, the two approaches lead exactly to the same definitions.

3.3 Illustrative Example

In order to show the influence of the choice of the conjunctions, t-conorms, implications, we illustrate a few operations on a one-dimensional example in Figure 1. Dilation, erosion and opening are performed using different operators. When using adjoint operators, opening is a "true" opening (i.e. increasing, anti-extensive and idempotent). It is clear in this figure that when using min and max for instance, which are dual but not adjoint, opening is not anti-extensive (it is not idempotent either, but it is increasing). On the contrary, using Kleene-Dienes adjoint operators, the opening is anti-extensive (Figure 1 d). However, other properties of erosion and dilation are lost, due to the weaker properties of the conjunction with respect to the ones of t-norms. These aspects will be further investigated in Section 4. The results obtained with Lukasiewicz operators are in this case very close to the original fuzzy set. However, all properties of all operations hold when using these operators.

4 General Forms of Fuzzy Morphological Dilation and Erosion

The second main result of this paper establishes the general form of fuzzy dilation and erosion, in order to satisfy a set of properties. Let $\delta_\nu(\mu)$ be a morphological dilation. Let us consider the following general form of δ:

$$\delta_\nu(\mu)(x) = g(f(\nu(x-y), \mu(y)), y \in \mathcal{S}), \tag{8}$$

where f is a mapping from $[0,1] \times [0,1]$ in $[0,1]$ and g is a mapping from $[0,1]^{\mathcal{S}}$ into $[0,1]$ (the result is then a fuzzy set).

Theorem 2. *The compatibility of fuzzy dilation with classical dilation in case ν is crisp, its increasingness, and the commutativity with the supremum lead to the only possible form of δ:*

$$\delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} t(\nu(x-y), \mu(y))$$

where t is a conjunction. If the commutativity ($\delta_\nu(\mu) = \delta_\mu(\nu)$) and iterativity ($\delta_\nu \delta_{\nu'}(\mu) = \delta_{\delta_\nu(\nu')}(\mu)$) properties are also required, then t has to be a t-norm.

From this dilation, a unique erosion such that $(\varepsilon_\nu, \delta_\nu)$ is an adjunction is derived:

$$\varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} I(\nu(y - x), \mu(y)),$$

where I is the adjoint of t.

If duality is required, $\hat{I}$ has to be the dual of t.

Proof. Let g_1 be the version of g applying on one variable only. Most results are derived by considering constant membership functions. Increasingness of δ implies that $g_1.f$ should be increasing in μ and ν. If ν is crisp, the compatibility with classical dilation implies that $\forall a \in [0,1], g_1.f(1, a) = a$. Therefore $g_1.f$ is a conjunction.

Further properties such as commutativity and iterativity imply $g_1.f$ be commutative and associative, respectively, i.e. it should be a t-norm.

It is easy to prove that g_1 has to be a bijection (one to one mapping). It follows that $f(1, a) = g_1^{-1}(a)$. Let $\mu'(y) = g_1^{-1}(\mu(y))$. The compatibility with classical morphology implies $\sup_{y \in \mathcal{S}} \mu(y) = g(g_1^{-1}(\mu(y)), y \in \mathcal{S})$, i.e. $\sup_{y \in \mathcal{S}} g_1(\mu'(y)) = g(\mu'(y), y \in \mathcal{S})$. Therefore $\delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} g_1.f(\nu(x - y), \mu(y))$. From the properties of t-norms, this form commutes with the supremum.

From a dilation δ_ν, a general result on adjunctions guarantees that there exists a unique erosion ε_ν such that $(\varepsilon_\nu, \delta_\nu)$ is an adjunction, and it is given by:

$$\varepsilon_\nu(\mu) = \bigvee \{\mu', \delta_\nu(\mu') \leq \mu\}.$$

We have the following equivalences, by denoting $g_1.f = t$ and I the adjoint of t:

$$\begin{aligned}
\delta_\nu(\mu') \leq \mu &\Leftrightarrow \forall x \in \mathcal{S}, \delta_\nu(\mu')(x) \leq \mu(x) \\
&\Leftrightarrow \forall x, y \in \mathcal{S}, t(\nu(x - y), \mu'(y)) \leq \mu(x) \\
&\Leftrightarrow \forall x, y \in \mathcal{S}, \mu'(y) \leq I(\nu(x - y), \mu(x)) \\
&\Leftrightarrow \forall y \in \mathcal{S}, \mu'(y) \leq \inf_{x \in \mathcal{S}} I(\nu(x - y), \mu(x))
\end{aligned}$$

Since ε_ν is the supremum of μ' verifying this equation, we have: $\varepsilon_\nu(\mu)(y) = \inf_{x \in \mathcal{S}} I(\nu(x - y), \mu(x))$.

Now, if duality is required between ε_ν and δ_ν with respect to complementation, it is straightforward to show that t and $\hat{I}$ have to be dual operators.

Having both duality and adjunction is possible under the conditions expressed in Theorem 1. ■

In [3], a similar approach was developed for deriving a general form of fuzzy inclusion (from which fuzzy erosion is derived). Since weaker properties are required, this approach leads to the use of weak t-norms and t-conorms (they are not associative and do not admit 1 (respectively 0) as unit element, in general). Properties of morphological operators are then weaker (no iterativity can be expected, no compatibility with classical morphology), and this is therefore somewhat less interesting from a morphological point of view. Our approach overcomes these drawbacks.

5 Conclusion

This paper exhibits the exact conditions to have a convergence between the two main approaches for fuzzy morphology. Although the underlying principles are not compatible in general, it is interesting to note that in case they are consistent, then both approaches are equivalent. Furthermore, they provide the most general forms in order to satisfy a set of reasonable properties as in classical morphology. These two new results clarify the status of different forms of mathematical morphology.

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