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Logic for physical space

From antiquity to present days

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Abstract Since the early days of physics, space has called for means to represent, experiment, and reason about it. Apart from physicists, the concept of space has intrigued also philosophers, mathematicians and, more recently, computer scientists. This longstanding interest has left us with a plethora of mathematical tools developed to represent and work with space. Here we take a special look at this evolution by considering the perspective of Logic. From the initial axiomatic efforts of Euclid, we revisit the major milestones in the logical representation of space and investigate current trends. In doing so, we do not only consider classical logic, but we indulge ourselves with modal logics. These present themselves naturally by providing simple axiomatizations of different geometries, topologies, space-time causality, and vector spaces.

Keywords Modal logic · Geometry · Topology · Mathematical morphology

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1 Context

Throughout the centuries, physics has evolved in its way of looking at space, Jammer (1993). From the initial tools necessary to measure the earth—geometry in the literal sense—modern physics is now inventing tools for understanding the universe both at microscopic and macroscopic granularities. Recently discovered phenomena such as quantum-mechanical superposition (Loll et al. (2006)) and space-time causality need to be taken into account.[1] But where did it all begin? Apparently the need of understanding space and laying down the foundations of how to represent it is quite old. If today we are considering how one can find a logical theory of quantum-mechanical superposition, it is because we have started long ago trying to define a theory for geometry, as exemplified by Euclid's success. In other words, it appears only natural to pursue logical axiomatizations of space if we want to be able to reason and compute quantities tied to it.

The present article does not claim to be a review of spatial logics—for which we refer to Aiello et al. (2007)—rather it aims at offering a few highlights on the foundations and recent developments in logic and space. The links with physics, some already explored, some entirely open to investigation, are the leitmotiv of the current presentation. But let us start by considering where it all may have started.

After an historical note about axiomatic geometry, we move to an insight on the developments of modal logics of topology, and finally provide examples of specific modal logics dealing with spatial domains and vector spaces.

2 Axiomatic, logical, and model-theoretic treatments of geometry

Not surprisingly, the first ever systematic (in today's terminology *axiomatic*) development of a mathematical discipline was Euclid's *Elements* (see, e.g., Kline (1972); Stillwell (2004, 2005); Hartshorne (1997) for historical accounts and modern expositions). Along with the notion of number, the basic geometric concepts of point, line, circle, etc. are some of the most natural abstractions that *Homo Sapiens* has derived from the surrounding physical world, and has been trying to understand since the beginning of its contemplation of the world. Euclid's seminal contribution, however, was his novel treatment of geometry, not as a set of empirical observations and practical methods for measuring distances, area of land, etc., but as an abstract mathematical theory, which, while rooted in the perceived reality, had nevertheless its own, absolute right of existence and development. And, it must have been a true stroke of a genius that he formulated the celebrated fifth postulate[2] as such, rather than coming up with a 'proof' of it. After centuries of numerous futile and wrong attempts to prove the postulate based on the others in Euclid's 'axiomatic system', only in the beginning of the nineteenth century the mathematical community reached the level of maturity

[1] See the Logic and Relativity article in this issue from the Hungarian school.

[2] If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles.

to assume, and soon afterwards prove, that it is conceivable that physical space may actually not satisfy Euclid's fifth postulate. That lead to the birth of non-Euclidean geometries by Bolyai, Lobachevsky, Gauss, and others (see, e.g., Coxeter (1969); Eves (1972); Meserve (1983)), who proved mathematically that each of these was equally *logically consistent* with the classical Euclidean model of space, considered then to be the one and only true abstraction of physical space. Only a century later came a confirmation from Einstein's relativity theory that some of these, inconceivable by the immediate senses, 'exotic' models of geometric space not only have their right of logical existence, but may actually be *the true model* of the physical (at least, cosmological) space. That discovery was an unsurpassed manifestation of the superiority of the abstract, logical approach of mathematics over the empirical approach underpinning the natural sciences, at least when it comes to comprehending such fundamental physical concepts as space and time.

The discovery of non-Euclidean geometries was essentially the first application of logical ideas to the study of space. By legitimizing the pluralism of possible geometries, it gradually shifted the essence of geometry from study of (physical) space, to study of *models* of (physical) space, and eventually to study of *models of logical theories* of (physical) space, thus placing it in the scope of *model theory*—a branch of mathematical logic studying the interplay between logic and mathematics. It was shown by Bolyai, Lobachevsky, Euler, Poincaré, and others that, depending on the acceptance, rejection, or replacement of Euclid's fifth postulate by suitable alternatives, a variety of natural alternative geometries evolve, incl., *hyperbolic* and *elliptic* geometries, as well as *absolute*, *affine*, and *projective* geometries (see, e.g., Coxeter (1969)). Each of these geometries can lay claim to model essential aspects and fragments of the real, physical space, without pretending to capture it in any absolute sense, nor in its entirety.

The axiomatic approach to geometry, conceived in Euclid's work, and invigorated by the emergence of non-Euclidean geometries, was only developed further at the end of the nineteenth century, by a surge of axiomatic investigations of the foundations of geometry by Peano, Pieri, Veblen, Pasch, Hilbert, and others, who analyzed various axiomatic systems and the mutual relationships between the primitive notions of Euclidean geometry. Since then, that trend has gradually abated, with the remarkable exception of Tarski and his school (see further), and in more recent times it has been followed by few researchers, notably in the sustained work of Pambuccian on axiomatic aspects of Euclidean, hyperbolic, absolute, and other geometries; see, e.g., Pambuccian (1989, 2001a,b, 2004).

While Euclid's work conceived the axiomatic idea, it (naturally) did not meet the modern-day standards for logical and mathematical rigour. Only in the beginning of the twentieth century, David Hilbert—one of the most influential mathematicians of all times and the strongest proponent of the axiomatic method in mathematics—recast Euclid's work into a precise and rigorous modern treatise, Hilbert (1950), which eventually put geometry on sound axiomatic foundations.

Another, initially unrelated, line of historical development was the *analytic method* in geometry of space (see e.g, Stillwell (2004, 2005); Hartshorne (1997) for concise popular expositions), going back to the *coordinatization* of the Euclidean plane and space in the first half of the seventeenth century by René Descartes, who introduced coordinate systems, thus enabling the solution of purely geometric problems by using purely algebraic methods. Later the algebraic perspective led to the emergence of the modern view on geometry demonstrated in Klein's *Erlangen program*, defining geometry as a study *not of figures, but of transformations*, and classifying different geometric structures and their theories not in terms of their shape or size, but rather by means of the groups of transformations which preserve them, thus placing geometry firmly on the abstract algebraic foundations.

The axiomatic method in geometry reached its logical maturity and met the algebraic and model-theoretic approaches in the seminal work of Alfred Tarski and his students and followers in the 1920–1970's (see Tarski (1959, 1967); Schwabhäuser et al. (1983); Tarski and Givant (1999)). Tarski (1959) developed systematically the logical foundations of *elementary geometry*, as "*that part of Euclidean geometry that can be formulated and established without the help of any set-theoretical devices*". Essentially, by "elementary geometry", Tarski meant the elementary (i.e., first-order) theory of Euclidean geometry, developed over a suitably expressive first-order language. In particular, Tarski showed (following Veblen's idea) that the whole of elementary geometry can be developed axiomatically using just two geometric relations, viz. *betweenness* and *equidistance*. He also demonstrated how the elementary geometry of the real plane can be formally interpreted in the elementary theory of the real-closed fields, which is the same as the elementary theory of the field of reals, by exploiting the coordinatization of the Euclidean plane and space. Furthermore, Tarski proved the completeness and decidability of the first-order theory of the field of reals, thus obtaining an explicit *decision procedure for the elementary geometry*; i.e., a general algorithmic method for deciding the truth of any statement in Euclidean geometry that can be translated to the first-order theory of the field of reals, by means of using a coordinate system in the plane (or in any finite-dimensional Euclidean space $\mathbb{R}^n$). The logical core of Tarski's decision method is his *quantifier elimination* procedure for the first-order language of the field of reals $\mathbb{R}$, which, applied to any given sentence in the first-order language of $\mathbb{R}$, produces a logically equivalent quantifier-free sentence; i.e., a Boolean combination of polynomial equations and inequalities. Subsets of the Euclidean space $\mathbb{R}^n$ definable by such formulas are called *semi-algebraic sets*. In particular, Tarski's result implies that the (parametrically) first-order definable relations in $\mathbb{R}^n$ are precisely the semi-algebraic sets of $\mathbb{R}^n$. For a sketch of an algebraic proof of this result, based on Sturm's theorem; see, e.g., Hodges (1993).

Tarski's original decision procedure is practically inefficient as it has non-elementary complexity. More efficient decision procedures were developed later by Monk, Solovay[3], Seidenberg, Collins, and others. Currently there are several well-developed and applied automated theorem proving decision methods for the first-order theory of the field of reals, and in particular, for the elementary geometry. Probably the practically most popular decision method for the theory of real closed fields, and the first one amenable to practical automation (and, in fact, implemented), was Collins' method of *Cylindrical Algebraic Decompositions* (CAD), based on an improved version of Tarski's quantifier elimination (see Collins (1998) for a recent survey on CAD).

[3] Monk and Solovay found a triple exponential algorithm, later improved to double exponential by Solovay; see, e.g., Feferman (2006).

While being a dramatic improvement of Tarski's original procedure, Collins' CAD algorithm is not the most efficient one. Currently some of the most efficient algorithms for quantifier elimination can be found in Basu et al. (1996) and Basu (1999). Other methods for automated reasoning in geometry include the *Characteristic Set Method* of Ritt and Wu (Chou (1988)) and the *Gröbner Basis Method* of Buchberger et al. (1988). For an overview of automated reasoning in geometry see Chou and Gao (2001) and for a detailed discussion on logical theories of fragments of elementary geometry see Balbiani et al. (2007).

Coming to the present days, the impact of contemporary mathematical logic on geometry of space is not so much by means of deductive systems and automated reasoning, but mainly through model theory. One of the main problems studied by classical model theory is the *logical definability* of functions and properties (sets and relations) in a given mathematical structure by means of formulas in a suitable (first-order) logical language. In particular, the characterization and study of sets and relations in the field of real numbers definable in first-order logic has turned out to be of fundamental importance for some of the main geometric applications of first-order logic. Indeed, Tarski's quantifier elimination and decision procedure for $\mathbb{R}^n$ were closely related to the study of the semi-algebraic sets as the first-order definable subsets of $\mathbb{R}^n$. Likewise important and fruitful has been the study of *constructible sets* in *algebraic geometry*. In turn, these ideas have led to the recent model-theoretic study of *order-minimal* (also known as *o-minimal*) *structures*—ordered structures in which every first-order definable subset is a union of finitely many points and open intervals. These structures share many good geometric properties with $\mathbb{R}^n$. Thus, the study of o-minimal structures deeply generalizes real algebraic geometry. In particular, the recent discoveries that the extensions of the real field with exponentiation and with analytic functions restricted to the unit hypercube are o-minimal lead to even stronger and deeper applications of mathematical logic to geometric theory of space. For further details see Haskell et al. (2000) and Macintyre (2003).

Another promising line of recent applications of logic to formal modeling, analysis, and reasoning about space is through *non-classical logic* in general, and through *modal logic* in particular. A rich variety of modal logics have recently been proposed as alternative logical languages to first-order logic, because of their simpler syntax and semantics. Moreover, unlike the general purpose first-order languages, modal logics are more suited for specific applications, and most importantly have better computational behavior: very often they are decidable, in PSPACE or at most EXPTIME, as opposed to the usually undecidable first-order counterparts. In particular, modal logics for parallelism, orthogonality, incidence, affine, and projective geometries have been introduced and studied in Balbiani et al. (1997); Balbiani (1998); Venema (1999); Balbiani and Goranko (2002); for further details see Balbiani et al. (2007). Other applications of modal logic to the study of space will be discussed in the next sections.

3 Modal logics of topology

The formalization of the concept of distance was one of the major goals of mathematicians throughout the centuries. A modern way to formalize it is through *metric*.

A further generalization of metric leads to the concept of *neighborhood*, which gives rise to *topology*—a branch of mathematics formed in the late nineteen/early twentieth century, see e.g., Singer and Thorpe (1967). Alongside algebra and geometry, topology became one of the fundamental branches of contemporary mathematics. Logicians started to explore many exciting connections between logic and topology soon after the emergence of topology. Already in 1938 Tarski showed that interpreting formulas of intuitionistic logic, as open subsets of a topological space gives an adequate semantics of intuitionistic logic—one of the first completeness results for intuitionistic logic, Tarski (1938). In the mid forties of the twentieth century, Tarski in collaboration with McKinsey gave the first topological interpretation of modalities by interpreting $\Box$ as the interior operator and $\Diamond$ as the closure operator of a topological space. As a result, they showed that the modal system S4 is sound and complete with respect to topological semantics, McKinsey and Tarski (1944). We recall that S4 is the fourth modal system in the list of eight modal systems introduced by Lewis back in 1918, which is considered as the beginning of the modern era of modal logic. It is axiomatized by postulating

$$\Box p \to p, \qquad \Box p \to \Box\Box p, \qquad \Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$$

as the axioms, and modus ponens ($\frac{\varphi \quad \varphi \to \psi}{\varphi}$) and necessitation ($\frac{\varphi}{\Box\varphi}$) as the rules of inference. The modal system S4 has a close link with intuitionistic logic (this can already be seen from Tarski's topological interpretations; Tarski (1938); McKinsey and Tarski (1944)), has a nice relational semantics (S4 is complete with respect to reflexive and transitive frames; see, e.g., Blackburn et al. (2001)), is considered as one of the basic systems for representing knowledge (read $\Box\varphi$ as 'an agent knows that φ'; Fagin et al. (1995)), and has a very attractive computational complexity (the decision problem of S4 is PSPACE-complete; see, e.g., Blackburn et al. (2001)). These alongside the above topological completeness, made S4 one of the most studied modal logics.

The topological completeness of S4 has two sides to it. We can view S4 as a simple decidable formalism to talk about topology, or we can view topological spaces as a nice mathematical tool to study S4 and its extensions.

In order to see how much of topology can be formalized in S4, we recall the celebrated result of McKinsey and Tarski (1944) that S4 is the modal logic of any Euclidean space. (In fact, the McKinsey-Tarski theorem is stronger than that, but this is all we need for our purposes.) This is a very exciting and nontrivial theorem. Its upside is that S4 is the logic of metrizable spaces, which capture the idea of distance (metric). The downside, though, is that such an important topological concept as being metrizable cannot be expressed in the basic modal language. In fact, neither are such important topological concepts as being connected, compact, Hausdorff, and the list goes on. A solution of this problem lies in extending the basic modal language $\mathcal{ML}$ by additional modalities which allow more expressive power. This, of course, has to be done carefully, because we would like to keep the resulting systems decidable, and even computationally attractive. This has recently been pursued by several authors. Shehtman enriched $\mathcal{ML}$ by the universal modality U (read $U\varphi$ iff φ is true everywhere in the model) and showed that in the enriched language the axiom

$$U(\Diamond p \to \Box p) \to (Up \vee U\neg p)$$

expresses connectedness, Shehtman (1999); Aiello and van Benthem (2002). Gabelaia showed that if we further enrich $\mathcal{ML}$ with the inequality modality $[\neq]$, then the lower separation axioms T_0 and T_1 also become expressible, Gabelaia (2001). Further results in this direction (including the extension of $\mathcal{ML}$ by nominals) can be found in ten Cate et al. (2009).

In order to see how helpful topological semantics can be for studying the landscape of extensions of S4, we first note that topological semantics for S4 includes its relational semantics, which dominated the study of modal logic for decades. Indeed, relational frames for S4 can be thought of as special topological spaces (see, e.g., Aiello et al. (2003)). In fact, topological semantics for S4 is more powerful than its relational semantics because, as follows from Gerson (1975), there exist extensions of S4 which are topologically complete, but relationally incomplete. In addition, most extensions of S4 that play an important role in the study of the structure of extensions of S4 turn out to be complete with respect to nice (classes of) topological spaces. A landscape of spatial logics over S4 can be found in van Benthem and Bezhanishvili (2007).

In order to establish interesting links with physics, we briefly recall that the modal system S4.2, which is the extension of S4 by the axiom $\Diamond\Box p \to \Box\Diamond p$, gives a modal axiomatization of the Minkowskian space-time, which provides the geometrical basis of Einstein's special relativity theory. More specifically, according to Minkowskian space-time, an event y comes after an event x if a signal can be sent from x to y at a speed not exceeding the speed of light. If we interpret $\Box$ as "it is the case now and it always will be the case that", then the resulting modal system is S4.2, Goldblatt (1980). Moreover, a possible future of our universe that the expansion will eventually force it to collapse to a singularity results in the modal system S4.1.2, which is the extension of S4.2 by the axiom $\Box\Diamond p \to \Diamond\Box p$. Further results on the axiomatization of modal logics of different 'natural' regions of Minkowskian space-time can be found in Goldblatt (1980); Shehtman (1983); Shapirovsky and Shehtman (2003, 2005).

The modal systems S4.2 and S4.1.2 play an important role in the interplay of modal logic and topology. Indeed, as follows from Gabelaia (2001), S4.2 defines the class of extremely disconnected spaces—an important class of topological spaces introduced by Stone back in 1937. One of the most important extremely disconnected spaces is the Stone-Čech compactification of the natural numbers, denoted by $\beta(\omega)$. It turns out that the modal logic of $\beta(\omega)$ is S4.1.2. More precisely, if $\omega^* = \beta(\omega) - \omega$ denotes the remainder of $\beta(\omega)$, then it is shown in Bezhanishvili and Harding (2009) that the modal logic of ω^* is S4 and that the modal logic of $\beta(\omega)$ is S4.1.2. It is interesting to point out that the proof involves an additional axiom of set theory which is not provable in ZFC (Zermelo-Fraenkel set theory with the Axiom of choice). It still remains an open problem whether the completeness of S4 with respect to ω^* and that of S4.1.2 with respect to $\beta(\omega)$ can be derived within ZFC.

Modal logic is also a useful tool in the study of *mathematical morphology*—a theory dedicated to the analysis of shape—and vector spaces, as we show next.

4 Vector spaces and mathematical morphology

Mathematical morphology (MM) is a theory dedicated to the analysis of shape, spatial information, and image processing. It was originally developed in the Paris School of Mines by Georges Matheron and Jean Serra. The first developments were carried out by Matheron while studying porous media (with applications to geostatistics), for which he developed a theory of shapes and random shapes, associated with measures based on topology and integral geometry, Matheron (1967, 1975). This naturally led to the early applications in image processing, first for binary images, then for grey-level and color images. Jean Serra (1982) provided an interesting optics point of view, which led to the study of invariance of operators under translation and scaling, and of properties such as local knowledge and (semi-)continuity. MM considers images containing geometrical shapes with luminance (or color) profiles, which can be investigated by their interactions with other shapes and luminance profiles. This makes the morphological approach especially relevant in the situations where image grey-levels (or colors) correspond directly to significant material data, as in medical imaging, microscopy, industrial inspection, and remote sensing.

MM relies on the concepts and tools from various branches of mathematics such as algebra, lattice theory, topology, discrete geometry, integral geometry, geometrical probability, and partial differential equations. In fact, any mathematical theory which deals with shapes, their combination and evolution can contribute to MM.

When adopting a logical point of view, the algebraic framework in which the basic structure is a complete lattice $(L, \leq)$ is particularly important and relevant. We denote the suprema and infima in L by $\bigvee$ and $\bigwedge$, respectively. A *dilation* is a unary operator $\delta : L \to L$ commuting with suprema, and an *erosion* is a unary operator $\varepsilon : L \to L$ commuting with infima; that is, $\delta(\bigvee_i x_i) = \bigvee_i \delta(x_i)$ and $\varepsilon(\bigwedge_i x_i) = \bigwedge_i \varepsilon(x_i)$ for each family (x_i) of elements of L (finite or not). Visually, one can think of a dilation of a shape as an enlargement of the shape. Dually, an erosion is a reduction of the shape in size. These are the two basic operators from which many others are built.

Another important concept is that of *adjunction*. A pair of operators (ε, δ) defines an adjunction on $(L, \leq)$ if for all $x, y \in L$ we have:

$$\delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y).$$

If a pair of operators (ε, δ) defines an adjunction, the following important properties hold:

(i) δ is a dilation and ε is an erosion;
(ii) $\delta\varepsilon \preceq Id$, where Id denotes the identity mapping on L, and $Id \preceq \varepsilon\delta$;
(iii) $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$ and $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$; that is, the compositions of a dilation and an erosion (defining morphological opening and closing) are idempotent operators.

The following representation theorem holds: an increasing operator δ is a dilation if $\exists\varepsilon$ such that (ε, δ) is an adjunction; the operator ε is then an erosion and $\varepsilon(x) = \bigvee\{y \in L : \delta(y) \leq x\}$. A similar result holds for erosion. Finally, let δ and ε be two increasing operators such that $\delta\varepsilon$ is anti-extensive and $\varepsilon\delta$ is extensive.[4] Then (ε, δ) is an adjunction (and hence δ is a dilation and ε is an erosion). More details on the algebraic framework can be found, e.g., in Heijmans and Ronse (1990); Ronse and Heijmans (1991).

In a spatial domain, for instance in the lattice $(\mathcal{F}(\mathbb{R}^n), \subseteq)$ of closed sets of $\mathbb{R}^n$, it is often useful to consider operators that are invariant under translation, in agreement with the optics point of view. In such cases, it can be shown that there exists a set B, called a *structuring element*, such that dilation and erosion are expressed as

$$\delta(X) = \{x \in \mathbb{R}^n : \check{B}_x \cap X \neq \emptyset\}$$

and

$$\varepsilon(X) = \{x \in \mathbb{R}^n : B_x \subseteq X\},$$

where B_x denotes the translation of B at a point x (that is, $x + B$), and $\check{B}$ denotes the operator symmetric to B with respect to the origin of space, Serra (1982). Similar results hold for the lattice of functions defined on a spatial domain, in which case we obtain nice physical interpretations. For instance, continuity expresses that if there are small changes in the spatial representation, then the result of a transformation undergoes only small changes; idempotence of filters such as openings, closings, and their compositions means that once some parts of objects or background have been filtered out, applying the same filter will have no more effect.[5]

The structuring element captures the notion of local information: the result of a transformation at point x depends on the information contained in a neighborhood of x defined by B.

This framework makes the theory applicable to many different contexts as soon as a lattice structure can be defined: sets and functions (the most classical use of MM relies on the lattice of powerset with set-theoretic inclusion, and on the lattice of functions with the usual partial order), logic, and (bipolar) fuzzy sets, Bloch and Maître (1995); Bloch (2006, 2007, 2011). Note that all these formal settings are interesting for dealing with physical space from either a quantitative point of view (sets and functions), a semi-quantitative point of view, taking into account spatial imprecision (fuzzy sets), or a qualitative and symbolic point of view (logic).

The idea of using MM in a logical framework was first introduced in Bloch and Lang (2000). Since knowing a formula is equivalent to knowing the set of its models, we can identify a formula φ with the set of its models $[\![\varphi]\!]$, and then apply set-theoretic morphological operations. Applying this idea to propositional logics has led to new tools for knowledge representation and reasoning, such as revision, fusion, abduction, and mediation Bloch and Lang (2000); Bloch et al. (2001, 2004, 2006b).

[4] An operator ψ is said to be anti-extensive if $\psi \leq Id$ where Id denotes the identity mapping, and extensive if $\psi \geq Id$.

[5] An operator ψ is idempotent if $\psi\psi = \psi$; that is, the composition of ψ with itself is equal to ψ, and thus applying ψ several times is the same as applying it only once.

By exploiting the strong similarity between the algebraic properties of MM operators and of modal operators, one can develop a modal morphologic, Bloch (2002).

In Sect. 3 we have recalled the interpretation of the S4 modalities $\Box$ and $\Diamond$ as topological interior and closure. The relational semantics interprets $\Box$ as necessity and $\Diamond$ as possibility on relational structures built of (possible) worlds linked by an accessibility relation; see, e.g., Blackburn et al. (2001). In MM, the key is to define the modal operators based on an accessibility relation as erosion and dilation:

$$\Box\varphi \equiv \varepsilon(\varphi) \quad \text{and} \quad \Diamond\varphi \equiv \delta(\varphi).$$

This can be done, as shown by Bloch (2002), by identifying the accessibility relation and the structuring element (i.e. $R(\omega, \omega')$ iff $\omega' \in B_\omega$), thus obtaining a modal logic built from morphological erosions and dilations which exhibits a number of interesting theorems and rules of inference. Interestingly enough, the MM framework endows the logic with additional properties that are useful for reasoning.

When dealing with physical spaces in general, and with vector spaces in particular, worlds can represent spatial entities, like spatial regions. Formulas then represent combinations of such entities. For instance, if a formula φ is a symbolic representation of a spatial region X, it can be interpreted as "the object we are looking at is in X." In an epistemic interpretation, it can represent the belief of an agent that the object is in X. Using these interpretations, if φ represents some knowledge or belief about a region X of the space, then $\Box\varphi$ represents a restriction of X: if we are looking at an object in X, then $\Box\varphi$ is a necessary region for this object. Similarly, $\Diamond\varphi$ represents an extension of X, and a possible region for the object. In an epistemic interpretation, $\Box\varphi$ represents the belief of an agent that the object is necessarily in the erosion of X, while $\Diamond\varphi$ represents the belief that it is possibly in the dilation of X.

For reasoning on vector spaces, and in particular for spatial reasoning, these modal logics are very efficient since several spatial relations can be formally modeled, including topological relations (e.g., adjacency, inclusion, part-whole, with nice links to other logics of space), distances, and directional relations, Bloch (2002); Bloch et al. (2006a). While the use of a structuring element allows reasoning on local spatial information, these models of spatial relations allow reasoning at a more structural level. For instance, two spatial entities are adjacent iff they are disjoint but the dilation of one of them meets the other; i.e., $\varphi \wedge \psi \vdash \bot$ and $\delta(\varphi) \wedge \psi \nvdash \bot$, with φ and ψ representing the two spatial entities. As another example, the minimal distance between two spatial entities can be expressed as $d(\varphi, \phi) = \min\{n : \delta^n(\varphi) \wedge \psi \nvdash \bot\}$, where δ^n denotes the dilation using as a structuring element a ball of a distance of radius n (defined in the spatial domain), or equivalently the composition of n dilations using a ball of radius 1. In a similar way, the Hausdorff distance can be derived from dilations. Such representations can be used in spatial reasoning, for instance to interpret a scene based on the structural arrangement of the objects it contains, Bloch (2006). For example, if a model of the scene is available, individual objects can be identified not only based on their intrinsic properties, but also by checking the relations they share with other objects. Spatial relations can also be used to guide the exploration of space, in a focus of attention process, and for recognition and interpretation tasks.

Another possibility is to look at MM as a set of operations in a vector space and from there consider a vector-like logic. *Arrow logic* is such a formalism. It is a form of modal logic where objects are transitions structured by various relations, rather than nodes in a labeled graph, Venema (1996). In particular, there is a binary modality for the composition of arrows and a unary modality for the inverse of an arrow. Such a language naturally models a vector space which, in turn, is the most intuitive underlying model of MM, Aiello and van Benthem (2002). In fact, arrow models consist of a ternary relation C, a binary relation R, and a unary relation I, which match the definition of a vector space:

(i) $(x, y, z) \in C$ iff $x = y + z$,
(ii) $(x, y) \in R$ iff $x = -y$,
(iii) $x \in I$ iff $x = e$,

where e is the identity vector and x, y, z are any vectors. The interpretation of a dilation operator in such a modal logic becomes the vector sum:

$$\{w : \exists v, v',\ w = v + v',\ v \in V(\phi), v' \in V(\psi)\} =$$
$$\{v + v' : v \in V(\phi), v' \in V(\psi)\},$$

where v, v', w are vectors and V is a valuation function from formulas to vectors.

A natural question here is how to axiomatize the resulting logic. We remark two issues. First, the axiom $x + (-x) = e$ poses a problem because it is not valid for arbitrary subsets of the universe; second, the operation $+$ must be defined for every pair of elements. These issues can be overcome by augmenting the traditional arrow logic with nominals (a technique coming from Hybrid Logic) and the universal modality (as we have seen in Sect. 3). This way we can define morphologic to be a hybrid arrow logic. The basic axiomatization can be found in de Freitas et al. (2002), though some interesting additions to capture MM are necessary, Aiello and Ottens (2007). The obtained logic is a powerful tool to describe shapes and reason about them. However, many fundamental questions remain unanswered:

(i) Is there a (convenient) finite axiomatization of arrow logic of the vector spaces $\mathbb{R}^n$?
(ii) Can the concept of dimension of a vector space be captured in arrow logic?
(iii) What about decidability and complexity of morphologics? Modal languages are usually well-behaved in this respect. However, it is likely that it is possible to encode the undecidable *tiling problem* in the arrow logic of the two-dimensional Euclidean plane.

5 Conclusions

Logic has influenced our study and understanding of the geometry of space in various ways, enriching and supplementing each other, including: methodologically, as an axiomatic method; mathematically, as a theory of models; and computationally, as methods and algorithms for automated reasoning. Each of these has hardly exhausted its potential, and we have all reasons to anticipate more powerful and exciting applications of logic to physics of space to be discovered in the time to come.

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