Marvels and mysteries of rational base numeration systems

Jacques Sakarovitch

CNRS / Université Paris-Diderot and Telecom Paris

University of Oxford— 19 June 2019

Based on the paper that introduced the rational base numeration systems:

- *Powers of rationals modulo 1 and rational base number systems*,
  Israel J. Math., 2008   with Sh. Akiyama and Ch. Frougny

and subsequent more recent works on the subject:

- *Trees and languages with periodic signature*,
  Indagationes Mathematicae 2017   with V. Marsault
- *On subtrees of the representation tree in rational base numeration systems*,
  DMTCS 2018   with Sh. Akiyama and V. Marsault

Outline of the talk

1. A problem by Mahler
2. Integer and Pisot base numeration systems
3. Representation of integers in a rational base
4. Representation of reals in a rational base
5. When order generates disorder
6. A property still missing a proper name (autosimilarity?)
7. Complements

Part I

A problem by Mahler

The fractional part of the powers of rational numbers

Notation

$\theta \in \mathbb{R}$ $\{\theta\}$ fractional part of θ

Theorem (Folk & Lore)

$\forall \theta \in \mathbb{Q}\,,$ *the sequence* $M(\theta) = \{n\theta\}$ *is finite.*

$\forall \theta \in \mathbb{R} \setminus \mathbb{Q}\,,$ *the sequence* $M(\theta)$ *is uniformly distributed.*

Problem

$\theta \in \mathbb{R}\,,\ \theta > 1$ Distribution of $S(\theta) = \big(\{\theta^n\}\big)_{n\in\mathbb{N}}$ **?**

Theorem (calculus classic)

For almost all θ, $S(\theta)$ *is uniformly distributed.*

The fractional part of the powers of rational numbers

Very few results are known for specific values of θ.

Proposition

θ *Pisot* $\implies$ 0 *is the only limit point of* $S(\theta)$ *(in* $\mathbb{R}/\mathbb{Z}$ *).*

Experimental results show that $S(\theta)$ looks :

- *uniformly distributed* for transcendental θ,
- *very chaotic* for rational θ.

Theorem (Pisot ?? — Vijayaraghavan 40)

θ *rational* $\implies$ $S(\theta)$ *has infinitely many limit points.*

Parametrization of the problem

Fix the rational $\frac{p}{q}$, $p > q \geqslant 2$ coprime integers.

New problem

$\xi \in \mathbb{R}$ Distribution of $M_{\frac{p}{q}}(\xi) = \left(\left\{\xi \left(\frac{p}{q}\right)^n\right\}\right)_{n\in\mathbb{N}}$ **?**

Theorem (the same as before)

For almost all ξ, $M_{\frac{p}{q}}(\xi)$ is uniformly distributed.

The (generalized) Mahler approach

Notation

$I \subsetneq [0,1[$ $\quad$ I will be a finite union of semi-closed intervals.

$$\mathbf{Z}_{\frac{p}{q}}(I) = \{\xi \in \mathbb{R} \mid \quad M_{\frac{p}{q}}(\xi) \quad \text{is eventually contained in } I \ \} \ .$$

Two directions of research:

Look for I as large as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is empty.

Look for I as small as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is non empty.

The (generalized) Mahler approach

Notation

$I \subsetneq [0,1[$ $\quad$ I will be a finite union of semi-closed intervals.

$$\mathbf{Z}_{\frac{p}{q}}(I) = \{\xi \in \mathbb{R} \;\big|\; M_{\frac{p}{q}}(\xi) \quad \text{is eventually contained in } I\ \} \ .$$

Two directions of research:

Look for I as large as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is empty.

Look for I as small as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is non empty.

Theorem (Mahler 68)

$\mathbf{Z}_{\frac{3}{2}}\left([0, \frac{1}{2}[\right)$ *is at most countable.*

The (generalized) Mahler approach

Notation

$I \subsetneq [0,1[$ $\quad$ I will be a finite union of semi-closed intervals.

$$\mathbf{Z}_{\frac{p}{q}}(I) = \{\xi \in \mathbb{R} \mid \quad M_{\frac{p}{q}}(\xi) \quad \text{is eventually contained in } I \ \} \ .$$

Two directions of research:

Look for I as large as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is empty.

Look for I as small as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is non empty.

Theorem (Mahler 68)

$\mathbf{Z}_{\frac{3}{2}}\left([0, \frac{1}{2}[\right)$ *is at most countable.*

Open problem $\qquad$ Is $\mathbf{Z}_{\frac{3}{2}}\left([0, \frac{1}{2}[\right)$ non empty?

The (generalized) Mahler approach

Notation

$I \subsetneq [0,1[$ $\quad$ I will be a finite union of semi-closed intervals.

$$\mathbf{Z}_{\frac{p}{q}}(I) = \{\xi \in \mathbb{R} \mid \quad M_{\frac{p}{q}}(\xi) \quad \text{is eventually contained in } I\ \}\ .$$

Two directions of research:

Look for I as large as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is empty.

Look for I as small as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is non empty.

Theorem (Mahler 68)

$\mathbf{Z}_{\frac{3}{2}}\left([0,\frac{1}{2}[\right)$ *is at most countable.*

Conjecture $\quad$ $\mathbf{Z}_{\frac{3}{2}}\left([0,\frac{1}{2}[\right)$ is empty.

The search for big I with empty $\mathbf{Z}_{\frac{p}{q}}(I)$

The search for big I with empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Flatto, Lagarias, Pollington 95)

The set of reals s

such that $\mathbf{Z}_{\frac{p}{q}}\left([s, s+\frac{1}{p}[\right)$ *is empty*

is dense in $[0, 1-\frac{1}{p}]$.

Theorem (Bugeaud 04)

The same set is of Lebesgue measure $1-\frac{1}{p}$.

The search for big I with empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Flatto, Lagarias, Pollington 95)

The set of reals s

such that $\mathbf{Z}_{\frac{p}{q}}\left([s, s+\frac{1}{p}[\right)$ *is empty*

is dense in $[0, 1-\frac{1}{p}]$.

Theorem (Bugeaud 04)

The same set is of Lebesgue measure $1-\frac{1}{p}$.

Conjecture — The same set is $[1-\frac{1}{p}]$.

The search for small l with non empty $\mathbf{Z}_{\frac{p}{q}}(l)$

The search for small I with non empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Pollington 81)

$$\mathbf{Z}_{\frac{3}{2}}\left(\left[\tfrac{4}{65}, \tfrac{61}{65}\right[\right) \quad \textit{is non empty.}$$

The search for small I with non empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Pollington 81)

$$\mathbf{Z}_{\frac{3}{2}}\left([\tfrac{4}{65},\tfrac{61}{65}[\right) \quad \textit{is non empty.}$$

Theorem (A.-F.-S. 05)

Let $p \geqslant 2q-1$. *There exists* $Y_{\frac{p}{q}} \subset [0,1[$ *of measure* $\frac{q}{p}$

such that $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ *is (countably) infinite.*

The search for small I with non empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Pollington 81)

$$\mathbf{Z}_{\frac{3}{2}}\left([\tfrac{4}{65},\tfrac{61}{65}[\right) \quad \textit{is non empty.}$$

Theorem (A.-F.-S. 05)

Let $p \geqslant 2q-1$. *There exists* $Y_{\frac{p}{q}} \subset [0,1[$ *of measure* $\frac{q}{p}$

such that $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ *is (countably) infinite.*

Indeed $\quad \mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right) = \{\xi \in \mathbb{R}_+ \mid \xi \text{ has two } \frac{p}{q}\text{-expansions} \}$.

The search for small I with non empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Pollington 81)

$$\mathbf{Z}_{\frac{3}{2}}\left([\tfrac{4}{65}, \tfrac{61}{65}[\right) \quad \textit{is non empty.}$$

Theorem (A.-F.-S. 05)

Let $p \geqslant 2q-1$. *There exists* $Y_{\frac{p}{q}} \subset [0,1[$ *of measure* $\frac{q}{p}$ *such that* $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ *is (countably) infinite.*

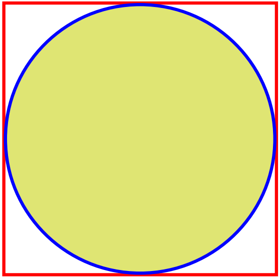
Indeed $\quad \mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right) = \{\xi \in \mathbb{R}_+ \mid \xi \text{ has two } \frac{p}{q}\text{-expansions } \}$.

What this means is what this talk is about.

Part II

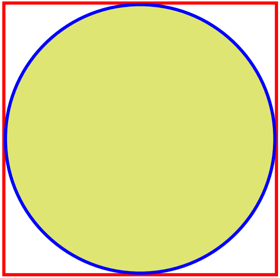
Integer and Pisot base numeration systems

Numbers do exist



$$\frac{\pi}{4} = \frac{C}{P} = \frac{D}{S}$$

Numbers do exist



But you have to write them in order to compute with them

Base 3 numeration system

$$N \in \mathbb{N}$$

Representation of N in base 3 : *word* in $A_3 = \{0, 1, 2\}^*$

Base 3 numeration system

$$N \in \mathbb{N}$$

Representation of N in base 3 : *word* in $A_3 = \{0, 1, 2\}^*$

$$\langle N \rangle_3 = a_k\, a_{k-1} \ldots a_1\, a_0 \qquad N = \sum_0^k a_i\, 3^i$$

Base 3 numeration system

$$N \in \mathbb{N}$$

Representation of N in base 3 : *word* in $A_3 = \{0, 1, 2\}^*$

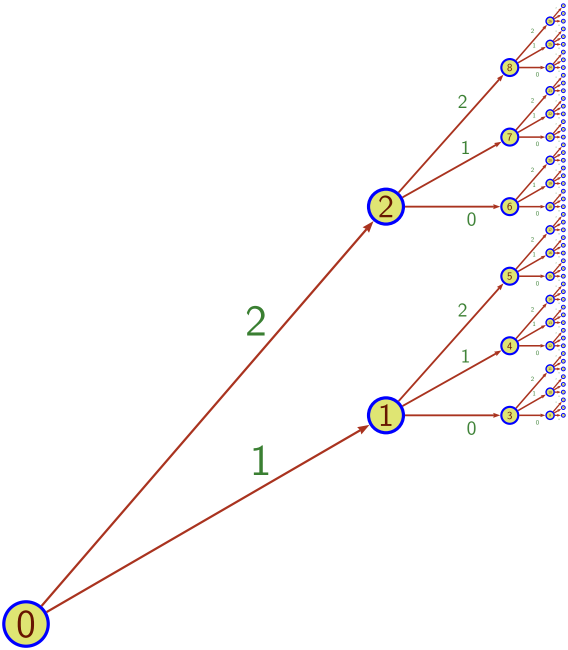
$$\langle N \rangle_3 = a_k \, a_{k-1} \dots a_1 \, a_0 \qquad N = \sum_0^k a_i \, 3^i$$

$$L_3 = \{ \langle N \rangle_3 \mid N \in \mathbb{N} \} = A_3^* \setminus 0 \, A_3^*$$

Base 3 numeration system

	0		111	13
1	1		112	14
2	2		120	15
10	3		121	16
11	4		122	17
12	5		200	18
20	6		201	19
21	7		202	20
22	8		210	21
100	9		211	22
101	10		212	23
102	11		220	24
110	12		221	25

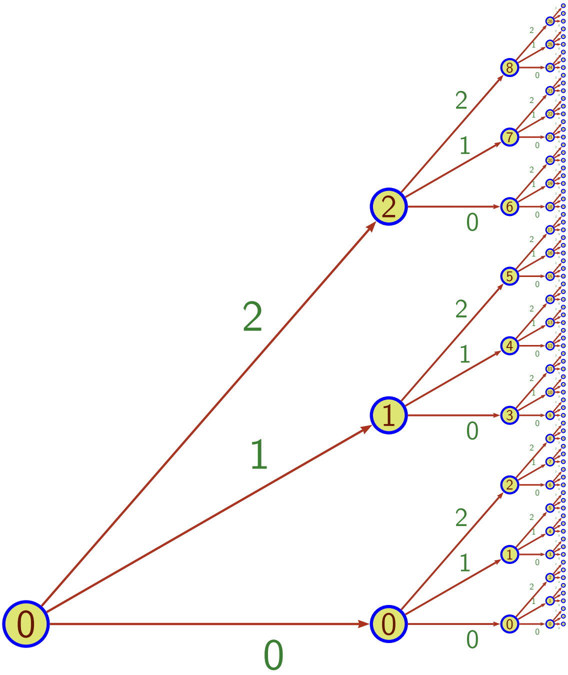
$$L_3 = \{\langle N \rangle_3 \mid N \in \mathbb{N}\} = A_3^* \setminus 0\,A_3^*$$


0
1
2
1
0
1
2
2
0
1
2
3
4
5
6
7
8

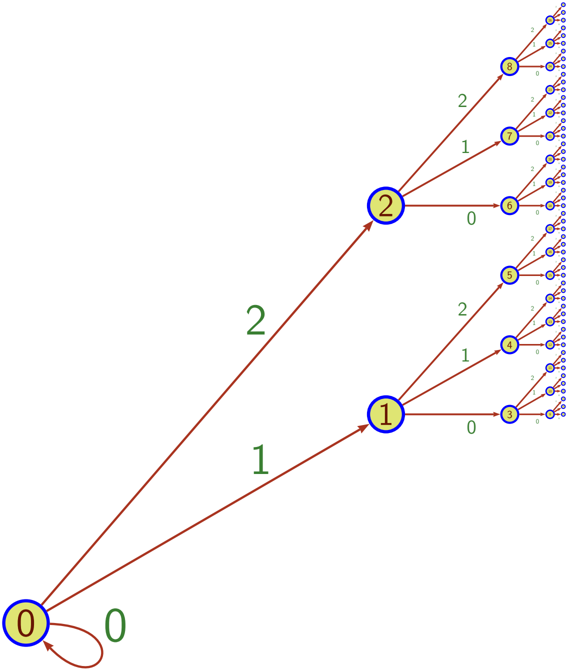
Base 3 numeration system

0000	0	0111	13
0001	1	0112	14
0002	2	0120	15
0010	3	0121	16
0011	4	0122	17
0012	5	0200	18
0020	6	0201	19
0021	7	0202	20
0022	8	0210	21
0100	9	0211	22
0101	10	0212	23
0102	11	0220	24
0110	12	0221	25

$$L'_3 = \{\langle N \rangle_3 \mid N \in \mathbb{N}\} = A_3^*$$



0
0
1
2
0
1
2
0
1
2
0
1
2
0
1
2
3
4
5
6
7
8



0
0
1
2
1
2
0
1
2
0
1
2
3
4
5
6
7
8

Computation of representations in base 3: the integers

$V = \left\{ v_i = (3)^i \;\middle|\; i \in \mathbb{N} \right\}$ together with $A_3 = \{0, 1, 2\}$

Greedy algorithm $N \in \mathbb{N}$ $\exists k$ $3^{k+1} > N \geqslant 3^k$

$$\begin{aligned} N_k &= N \\ N_{k-1} &= N_k - a_k\, 3^k & a_k \in A\,, \quad 3^k > N_{k-1} \\ N_{k-2} &= N_{k-1} - a_{k-1}\, 3^{k-1} & a_{k-1} \in A\,, \quad 3^{k-1} > N_{k-2} \\ &\dots & \dots \end{aligned}$$

$$N = \sum_0^k a_i\, 3^i \qquad \langle N \rangle_3 = a_k\, a_{k-1} \dots a_1\, a_0$$

Computation of representations in base 3: the integers

$V = \left\{ v_i = (3)^i \,\middle|\, i \in \mathbb{N} \right\}$ together with $A_3 = \{0, 1, 2\}$

Division algorithm $N \in \mathbb{N}$

$$
\begin{aligned}
N'_0 &= N \\
N'_0 &= 3\,N'_1 + a_0 && a_0 \in A \\
N'_1 &= 3\,N'_2 + a_1 && a_1 \in A \\
&\dots
\end{aligned}
$$

$$N = \sum_0^k a_i\,3^i \qquad \langle N \rangle_3 = a_k\,a_{k-1}\dots a_1\,a_0$$

Computation of representations in base 3: the integers

$V = \left\{ v_i = (3)^i \;\middle|\; i \in \mathbb{N} \right\}$ together with $A_3 = \{0, 1, 2\}$

Division algorithm $17 \in \mathbb{N}$

$$\begin{aligned} N'_0 &= 17 \\ 17 = N'_0 &= 3 \cdot 5 + 2 \qquad a_0 = 2 \in A \\ 5 = N'_1 &= 3 \cdot 1 + 2 \qquad a_1 = 2 \in A \\ 1 = N'_2 &= 3 \cdot 0 + 1 \qquad a_2 = 1 \in A \end{aligned}$$

$17 = ((1) \cdot 3 + 2) \cdot 3 + 2$ $\langle 17 \rangle_3 = 1\,2\,2$

Computation of representations in base 3: the reals

$V = \left\{ v_i = (3)^i \;\middle|\; i \in \mathbb{Z} \right\}$ together with $A_3 = \{0, 1, 2\}$

Computation of representations in base 3: the reals

$V = \left\{ v_i = (3)^i \;\middle|\; i \in \mathbb{Z} \right\}$ together with $A_3 = \{0, 1, 2\}$

Greedy algorithm $\quad x \in \mathbb{R} \quad \exists k \quad 3^{k+1} > x \geqslant 3^k$

$$x_k = x$$

$$x_{k-1} = x_k - a_k\, 3^k \qquad a_k \in A\,, \quad 3^k > x_{k-1}$$

$$x_{k-2} = x_{k-1} - a_{k-1}\, 3^{k-1} \qquad a_{k-1} \in A\,, \quad 3^{k-1} > x_{k-2}$$

$$\ldots \qquad \ldots$$

$$x = \sum_{-\infty}^{k} a_i\, 3^i \qquad \langle x \rangle_3 = a_k\, a_{k-1} \ldots a_1\, a_0 \,.\, a_{-1}\, a_{-2} \ldots$$

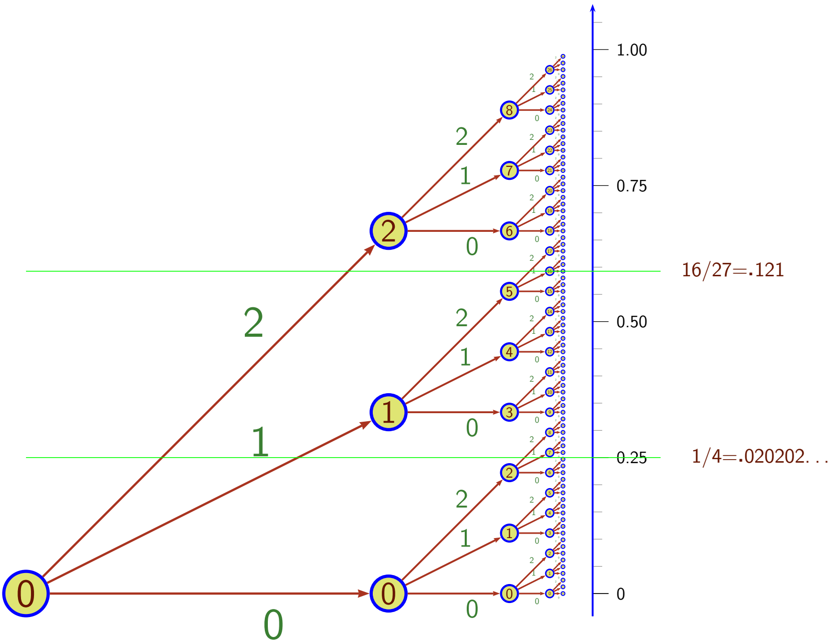
Computation of representations in base 3: the reals

$$V = \left\{ v_i = (3)^i \;\middle|\; i \in \mathbb{Z} \right\} \qquad \text{together with} \qquad A_3 = \{0, 1, 2\}$$

Greedy algorithm $\quad x \in [0, 1[$

$$x_1 = x \qquad a_i = \lfloor 3\, x_i \rfloor \qquad x_{i+1} = \{3\, x_i\}$$

$$x = \sum_{1}^{\infty} a_i\, 3^{-i} \qquad \langle x \rangle_3 = .\, a_1\, a_2\, a_3 \ldots$$


1.00
0.75
0.50
0.25
0
16/27=.121
1/4=.020202...

Integer base numeration systems and finite automata

Blaise Pascal   in *De numeris multiplicibus*   ∼1650

The p-representations of the integers divisible by k

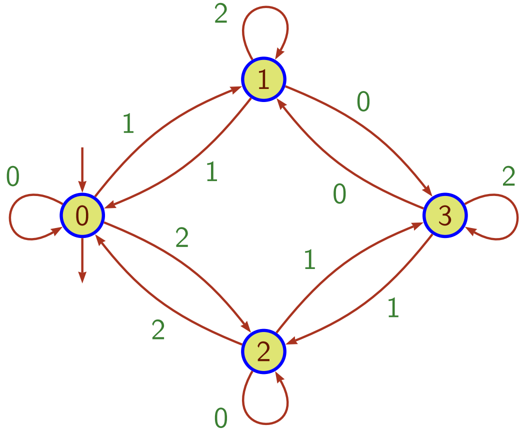
is recognised by a finite automaton.

Integer base numeration systems and finite automata

Blaise Pascal   in *De numeris multiplicibus*   ~1650

The p-representations of the integers divisible by k

is recognised by a finite automaton.



Normalisation in an integer base numeration system

Addition in base 3

	2	1	1	1	0
	2	0	1	2	1
1	1	2	0	0	1

Normalisation in an integer base numeration system

Addition in base 3

	2	1	1	1	0
	2	0	1	2	1
1	1	2	0	0	1

2	1	1	1	0
2	0	1	2	1
4	1	2	3	1

Normalisation in an integer base numeration system

Addition in base 3 is a normalisation

$$\begin{array}{cccccc} & 2 & 1 & 1 & 1 & 0 \\ & 2 & 0 & 1 & 2 & 1 \\ \hline 1 & 1 & 2 & 0 & 0 & 1 \end{array} \qquad \begin{array}{ccccc} 2 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 2 & 1 \\ \hline 4 & 1 & 2 & 3 & 1 \end{array}$$

$\nu\colon \{0,1,2,3,4\}^* \longrightarrow \{0,1,2\}^*$

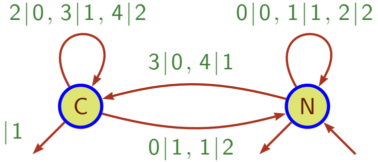
Normalisation in an integer base numeration system

Addition in base 3 is a normalisation

	2	1	1	1	0
	2	0	1	2	1
1	1	2	0	0	1

2	1	1	1	0
2	0	1	2	1
4	1	2	3	1

$\nu\colon \{0,1,2,3,4\}^* \longrightarrow \{0,1,2\}^*$



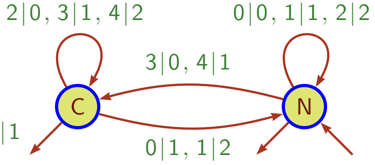
Normalisation in an integer base numeration system

Addition in base 3 is a normalisation

	2	1	1	1	0
	2	0	1	2	1
1	1	2	0	0	1

2	1	1	1	0
2	0	1	2	1
4	1	2	3	1

$$\nu \colon \{0,1,2,3,4\}^* \longrightarrow \{0,1,2\}^*$$



$$\underset{1}{\longleftarrow} \mathsf{C} \underset{1}{\overset{4}{\longleftarrow}} \mathsf{N} \underset{2}{\overset{1}{\longleftarrow}} \mathsf{C} \underset{0}{\overset{2}{\longleftarrow}} \mathsf{C} \underset{0}{\overset{3}{\longleftarrow}} \mathsf{N} \underset{1}{\overset{1}{\longleftarrow}} \mathsf{N} \longleftarrow$$

Normalisation in an integer base numeration system

Proposition (Folk & Lore)

In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter sequential right transducer.

Normalisation in an integer base numeration system

Proposition (Folk & Lore)

In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter sequential right transducer.

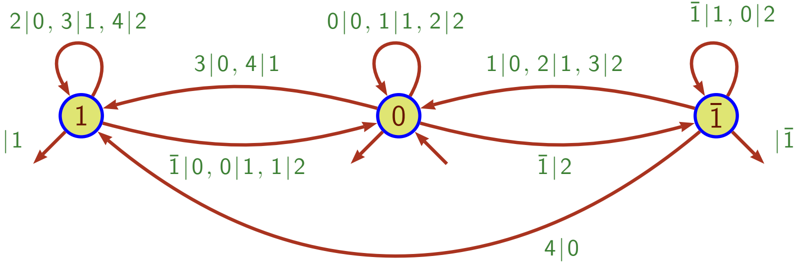
$$\nu \colon \{\bar{1}, 0, 1, 2, 3, 4\}^* \longrightarrow \{0, 1, 2\}^*$$

Normalisation in an integer base numeration system

Proposition (Folk & Lore)

In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter sequential right transducer.

$$\nu\colon \{\bar{1}, 0, 1, 2, 3, 4\}^* \longrightarrow \{0, 1, 2\}^*$$

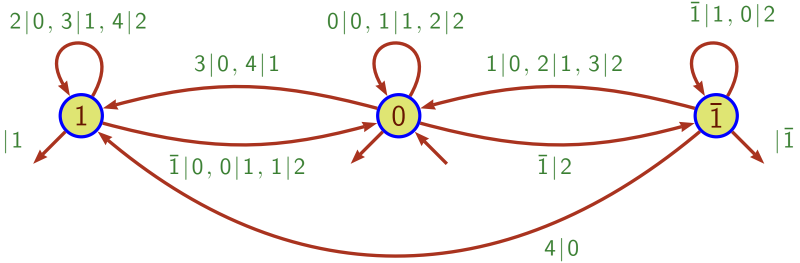


Normalisation in an integer base numeration system

Proposition (Folk & Lore)

In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter sequential right *transducer.*

$$\nu\colon \{\bar{1},0,1,2,3,4\}^* \longrightarrow \{0,1,2\}^*$$

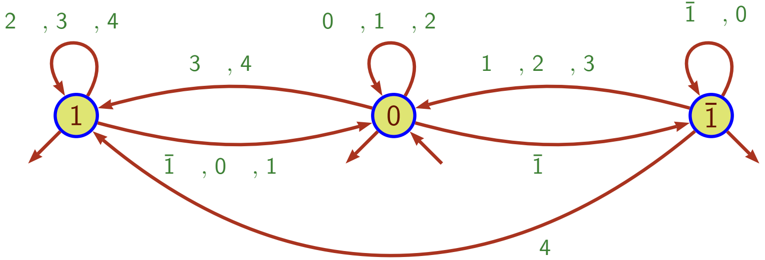


Normalisation in an integer base numeration system

Proposition (Folk & Lore)

In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter sequential right transducer.

$$\nu\colon \{\bar{1},0,1,2,3,4\}^* \longrightarrow \{0,1,2\}^*$$

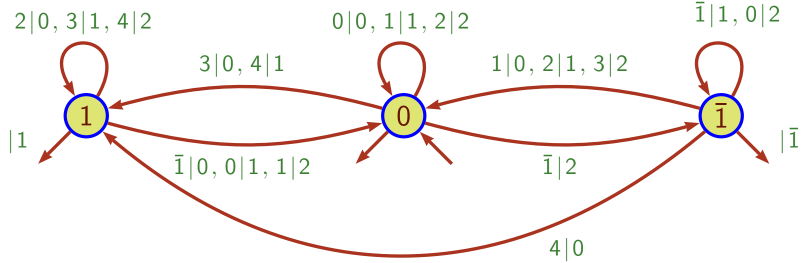


Normalisation in an integer base numeration system

Proposition (Folk & Lore)

In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter sequential right transducer.

$$\nu\colon \{\bar{1}, 0, 1, 2, 3, 4\}^* \longrightarrow \{0, 1, 2\}^*$$



The base β numeration system

3 an integer >1

$V = \left\{ v_i = (3)^i \,\middle|\, i \in \mathbb{Z} \right\}$ together with $A_3 = \{0, 1, 2\}$

The base β numeration system

β any real number > 1

$$V = \left\{ v_i = (\beta)^i \;\middle|\; i \in \mathbb{Z} \right\} \quad \text{together with} \quad A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$$

The base β numeration system

β any real number > 1

$V = \left\{ v_i = (\beta)^i \;\middle|\; i \in \mathbb{Z} \right\}$ together with $A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$

Greedy algorithm (Rényi 57)

$x \in [0, 1[$

$$x_1 = x \qquad a_i = \lfloor \beta \, x_i \rfloor \qquad x_{i+1} = \{ \beta \, x_i \}$$

$$x = \sum_{1}^{\infty} a_i \, \beta^{-i} \qquad \langle x \rangle_\beta = .\, a_1 \, a_2 \, a_3 \ldots$$

The base β numeration system

β any real number > 1

$V = \left\{ v_i = (\beta)^i \;\middle|\; i \in \mathbb{Z} \right\}$ together with $A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$

Theorem (Parry 60)

β *Pisot* $\Longrightarrow$ $L_\beta = \{\langle x \rangle_\beta \mid x \in \mathbb{R}\} \in \mathrm{Rat}\, A_\beta^{\mathbb{N}}$

The base β numeration system

β any real number > 1

$$V = \left\{ v_i = (\beta)^i \;\middle|\; i \in \mathbb{Z} \right\} \quad \text{together with} \quad A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$$

Theorem (Parry 60)

β *Pisot* $\implies$ $L_\beta = \{\langle x \rangle_\beta \mid x \in \mathbb{R}\} \in \mathrm{Rat}\, A_\beta^{\mathbb{N}}$

Theorem (Berend-Frougny 96)

β is Pisot iff in base β ,
normalisation from any alphabet of digits
is realised by a letter-to-letter (finite) transducer.

Part III

Representation of integers in a rational base

The base $\frac{3}{2}$ numeration system — the β numeration approach

$\frac{3}{2}$ a real number > 1

$V = \left\{ v_i = \left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{Z} \right\}$ together with $A_{\frac{3}{2}} = \{0, \ldots, \lceil \frac{3}{2} \rceil - 1\} = \{0, 1\}$

The base $\frac{3}{2}$ numeration system — the β numeration approach

$\frac{3}{2}$ a real number > 1 not a Pisot number

$V = \left\{ v_i = \left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{Z} \right\}$ together with $A_{\frac{3}{2}} = \{0, \ldots, \lceil \frac{3}{2} \rceil - 1\} = \{0, 1\}$

The base $\frac{3}{2}$ numeration system — the β numeration approach

$\frac{3}{2}$ a real number > 1 not a Pisot number

$V = \left\{ v_i = \left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{Z} \right\}$ together with $A_{\frac{3}{2}} = \{0, \ldots, \lceil \frac{3}{2} \rceil - 1\} = \{0, 1\}$

Greedy algorithm $x \in \mathbb{R}$ $\exists k$ $\left(\frac{3}{2}\right)^{k+1} > x \geqslant \left(\frac{3}{2}\right)^k$

$$x_k = x$$

$$x_{k-1} = x_k - a_k \left(\frac{3}{2}\right)^k \qquad a_k \in A\,, \quad \left(\frac{3}{2}\right)^k > x_{k-1}$$

$$x = \sum_{-\infty}^{k} a_i \left(\frac{3}{2}\right)^i \qquad \langle x \rangle_W = a_k\, a_{k-1} \ldots a_1\, a_0 . a_{-1}\, a_{-2} \ldots$$

$$\langle 2 \rangle_W = 10.010000010\ \ldots$$

The base $\frac{3}{2}$ numeration system — the Euclidean approach

$$\frac{3}{2} \qquad \text{a real number} \quad > 1$$

$$U = \left\{ u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A_3 = \{0, 1, 2\}$$

The base $\frac{3}{2}$ numeration system — the Euclidean approach

$$\frac{3}{2} \qquad \text{a real number} \quad > 1$$

$$U = \left\{ u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A_3 = \{0, 1, 2\}$$

Modified division algorithm $\qquad N \in \mathbb{N}$

$$\begin{aligned} N_0 &= N \\ 2\,N_0 &= 3\,N_1 + a_0 & a_0 \in A \\ 2\,N_1 &= 3\,N_2 + a_1 & a_1 \in A \\ &\dots \end{aligned}$$

$$N = \sum_0^k a_i \frac{1}{2}\left(\frac{3}{2}\right)^i \qquad \langle N \rangle_{\frac{3}{2}} = a_k\, a_{k-1} \dots a_1\, a_0$$

The base $\frac{3}{2}$ numeration system — the Euclidean approach

$$\frac{3}{2} \qquad \text{a real number} \quad > 1$$

$$U = \left\{ u_i = \frac{1}{2} \left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A_3 = \{0, 1, 2\}$$

Modified division algorithm $\qquad 5 \in \mathbb{N}$

$$N_0 = 5$$

$$2\,N_0 = 2 \cdot 5 = 3 \cdot 3 + 1 \qquad 1 \in A$$

$$2\,N_1 = 2 \cdot 3 = 3 \cdot 2 + 0 \qquad 0 \in A$$

$$2\,N_2 = 2 \cdot 2 = 3 \cdot 1 + 1 \qquad 1 \in A$$

$$2\,N_3 = 2 \cdot 1 = 3 \cdot 0 + 2 \qquad 2 \in A$$

$$5 = \frac{1}{2}\left[\left(\left((2) \cdot \frac{3}{2} + 1\right) \cdot \frac{3}{2} + 0\right) \cdot \frac{3}{2} + 1\right] \qquad \langle 5 \rangle_{\frac{3}{2}} = 2\,1\,0\,1$$

The base $\frac{3}{2}$ numeration system — the Euclidean approach

$$U = \left\{ u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A_3 = \{0, 1, 2\}$$

The base $\frac{3}{2}$ numeration system — the Euclidean approach

$$U = \left\{ u_i = \frac{1}{2} \left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A_3 = \{0, 1, 2\}$$

Theorem

Every N *in* $\mathbb{N}$ *has an integer representation in the* $\frac{3}{2}$*-system.*

It is the unique finite $\frac{3}{2}$*-representation of* N *.*

The base $\frac{3}{2}$ numeration system — the Euclidean approach

$$U = \left\{ u_i = \frac{1}{2} \left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A_3 = \{0, 1, 2\}$$

Theorem

Every N in $\mathbb{N}$ has an integer representation in the $\frac{3}{2}$-system.

It is the unique finite $\frac{3}{2}$-representation of N .

We call this representation the $\frac{3}{2}$-expansion of N

and we denote it by $\langle N \rangle_{\frac{3}{2}}$.

The base $\frac{3}{2}$ numeration system — the Euclidean approach

$$U = \left\{ u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A_3 = \{0, 1, 2\}$$

Theorem

Every N in $\mathbb{N}$ has an integer representation in the $\frac{3}{2}$-system.

It is the unique finite $\frac{3}{2}$-representation of N .

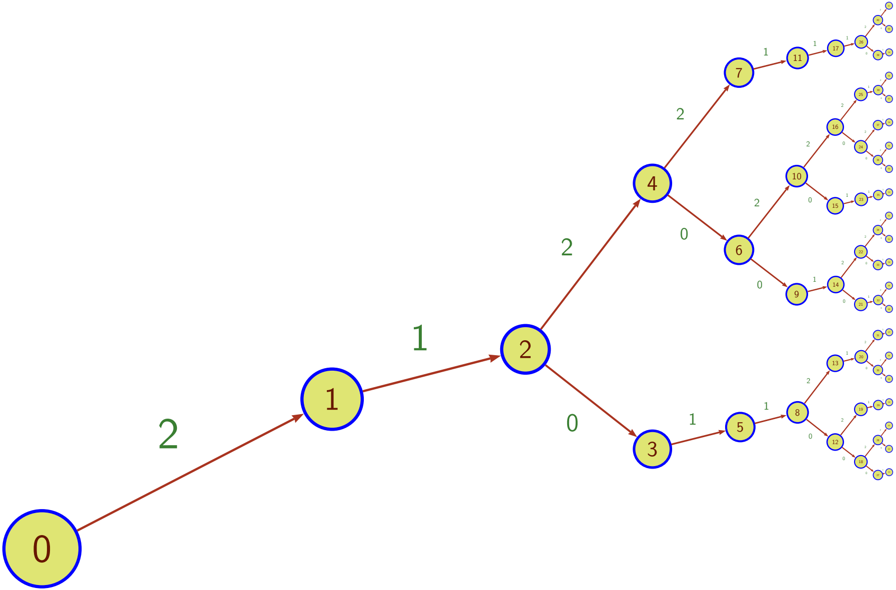
We call this representation the $\frac{3}{2}$-expansion of N

and we denote it by $\langle N \rangle_{\frac{3}{2}}$.

$$L_{\frac{3}{2}} = \left\{ \langle N \rangle_{\frac{3}{2}} \;\middle|\; N \in \mathbb{N} \right\} = ????$$

	0
2	1
21	2
210	3
212	4
2101	5
2120	6
2122	7
21011	8
21200	9
21202	10
21221	11
210110	12
210112	13
212001	14
212020	15
212022	16

212211	17
2101100	18
2101102	19
2101121	20
2120010	21
2120012	22
2120201	23
2120220	24
2120222	25
2122111	26
21011000	27
21011002	28
21011021	29
21011210	30
21011212	31
21200101	32
21200120	33


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17

The tree $T_{\frac{3}{2}}$ of the $\frac{3}{2}$-expansions

$L_{\frac{3}{2}}$ prefix-closed $\Longrightarrow$ $L_{\frac{3}{2}}$ spans the edges
of a subtree $T_{\frac{3}{2}}$ of the full 3-ary tree.

The nodes of $T_{\frac{3}{2}}$ are labeled by the integers.

The label of a node is the integer represented
by the label of the path from the root to that node.

These labels give *the radix order* in $L_{\frac{3}{2}}$.

The tree $T_{\frac{3}{2}}$ of the $\frac{3}{2}$-expansions

$L_{\frac{3}{2}}$ prefix-closed $\Longrightarrow$ $L_{\frac{3}{2}}$ spans the edges of a subtree $T_{\frac{3}{2}}$ of the full 3-ary tree.

The nodes of $T_{\frac{3}{2}}$ are labeled by the integers.

The label of a node is the integer represented by the label of the path from the root to that node.

These labels give *the radix order* in $L_{\frac{3}{2}}$.

Any two distinct subtrees of $T_{\frac{3}{2}}$ are not isomorphic.

The FLIP property

$$L \subseteq A^*$$

The FLIP property

$$L \subseteq A^*$$

Definition

L has the Finite Left Iteration Property (FLIP) if

$$\forall u, v \in A^* \qquad \left\{ i \in \mathbb{N} \,\middle|\, u\, v^i \in \mathrm{Pre}(L) \right\} \qquad \text{is finite.}$$

The FLIP property

$$L \subseteq A^*$$

Definition

L has the Finite Left Iteration Property (FLIP) if

$$\forall u, v \in A^* \qquad \left\{ i \in \mathbb{N} \,\middle|\, u\,v^i \in \mathrm{Pre}(L) \right\} \qquad \text{is finite.}$$

(Equivalently, if $\mathrm{Pre}(L)$ meets the *IRS* condition.)

The FLIP property

$$L \subseteq A^*$$

Definition

L has the Finite Left Iteration Property (FLIP) if

$$\forall u, v \in A^* \qquad \left\{ i \in \mathbb{N} \,\middle|\, u\, v^i \in \mathrm{Pre}(L) \right\} \qquad \text{is finite.}$$

(Equivalently, if $\mathrm{Pre}(L)$ meets the *IRS* condition.)

Proposition

$L_{\frac{p}{q}}$ *is a FLIP language.*

The FLIP property

$$L \subseteq A^*$$

Definition

L has the Finite Left Iteration Property (FLIP) if

$$\forall u, v \in A^* \qquad \left\{ i \in \mathbb{N} \,\middle|\, u\, v^i \in \mathrm{Pre}(L) \right\} \qquad \text{is finite.}$$

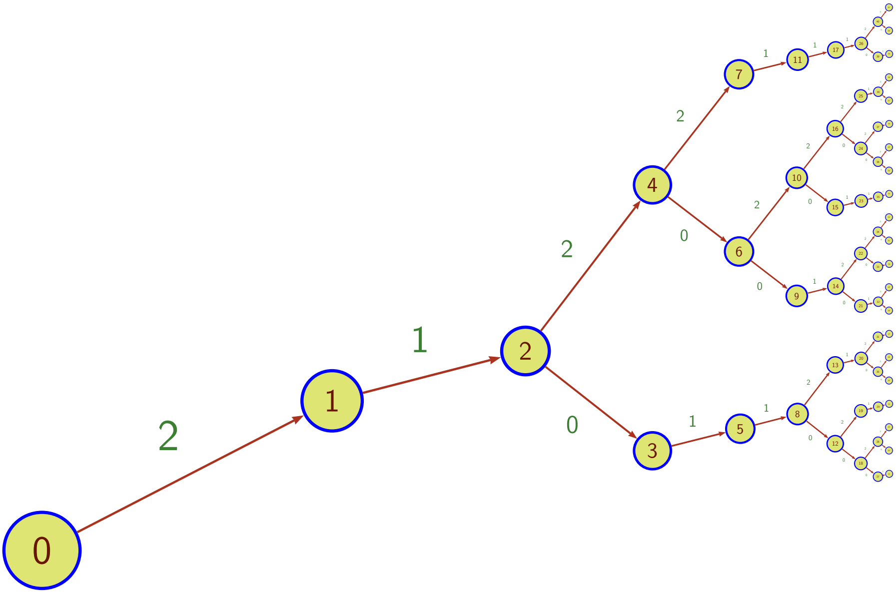
(Equivalently, if $\mathrm{Pre}(L)$ meets the *IRS* condition.)

Proposition

$L_{\frac{p}{q}}$ *is a FLIP language.*

Corollary

$L_{\frac{p}{q}}$ *is not a regular language, not a context-free language, not known to belong to any subclass of context-sensitive languages.*


0
2
1
1
2
2
0
3
0
1
5
4
2
0
7
6

Digit conversion

D finite digit alphabet, that contains A .

$$\chi_D \colon D^* \to A^* \qquad \forall w \in D^* \qquad \pi(\chi_D(w)) = \pi(w) \ .$$

Proposition

For every D , χ_D *is realised*
by a letter-to letter sequential right transducer.

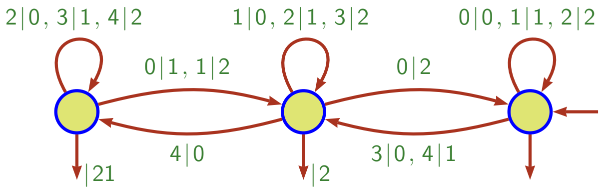
Digit conversion

D finite digit alphabet, that contains A .

$$\chi_D \colon D^* \to A^* \qquad \forall w \in D^* \qquad \pi(\chi_D(w)) = \pi(w) \ .$$

Proposition

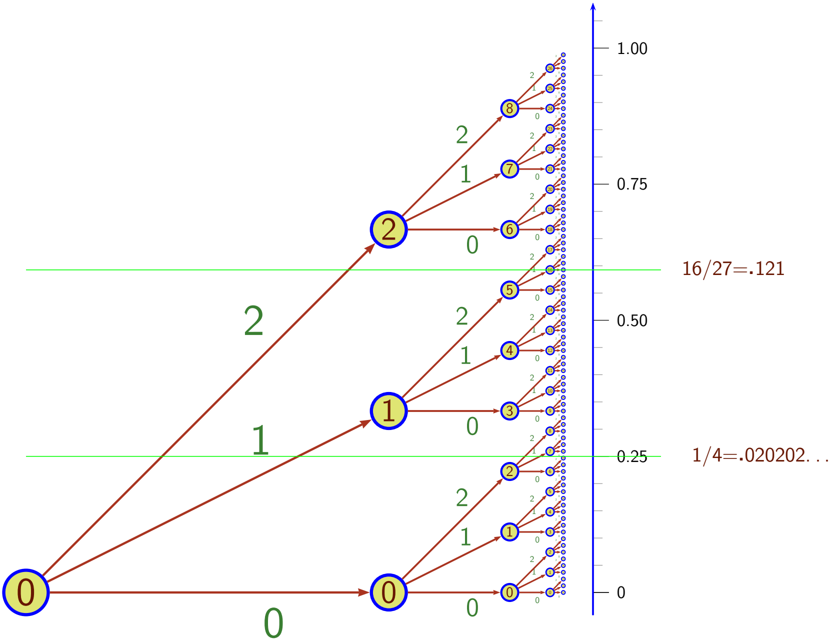
For every D , χ_D is realised by a letter-to letter sequential right transducer.



Part IV

Representation of reals in a rational base

Representation of reals in base 3: the tree T_3'



Representation of reals in base 3 : the tree T'_3

$A_3^{\mathbb{N}}$ = labels of the *infinite paths* in T'_3 $\qquad$ $\mathbf{a} = \{a_i\}_{i \geqslant 1} \in A_3^{\mathbb{N}}$

Definition

$\mathbf{a}$ is *an expansion in base* 3 of the real $x \in [0, 1]$ defined by:

$$x = \pi(.\mathbf{a}) = \sum_{i \geqslant 1} a_i \left(\frac{1}{3}\right)^i .$$

Representation of reals in base 3 : the tree T'_3

$A_3^{\mathbb{N}}$ = labels of the *infinite paths* in T'_3 $\quad$ $\mathbf{a} = \{a_i\}_{i \geqslant 1} \in A_3^{\mathbb{N}}$

Definition

$\mathbf{a}$ is *an expansion in base* 3 of the real $x \in [0, 1]$ defined by:

$$x = \pi(.\mathbf{a}) = \sum_{i \geqslant 1} a_i \left(\frac{1}{3}\right)^i .$$

Every real in $[0, 1]$ has (at least) one expansion in base 3 .

Representation of reals in base 3 : the tree T_3'

$A_3^{\mathbb{N}}$ = labels of the *infinite paths* in T_3' $\quad$ $\mathbf{a} = \{a_i\}_{i \geqslant 1} \in A_3^{\mathbb{N}}$

Definition

$\mathbf{a}$ is *an expansion in base* 3 of the real $x \in [0,1]$ defined by:

$$x = \pi(.\mathbf{a}) = \sum_{i \geqslant 1} a_i \left(\frac{1}{3}\right)^i .$$

Every real in $[0,1]$ has (at least) one expansion in base 3 .

Every real in $[0,1]$ has at most two expansions in base 3 .

Representation of reals in base 3 : the tree T'_3

$A_3^{\mathbb{N}}$ = labels of the *infinite paths* in T'_3 $\quad\quad$ $\mathbf{a} = \{a_i\}_{i \geqslant 1} \in A_3^{\mathbb{N}}$

Definition

$\mathbf{a}$ is *an expansion in base* 3 of the real $x \in [0,1]$ defined by:

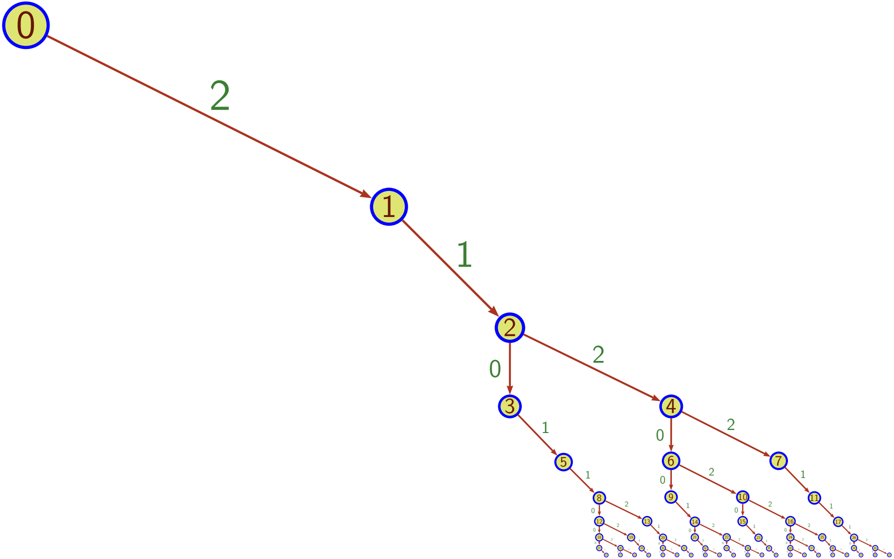
$$x = \pi(.\mathbf{a}) = \sum_{i \geqslant 1} a_i \left(\frac{1}{3}\right)^i .$$

Every real in $[0,1]$ has (at least) one expansion in base 3 .

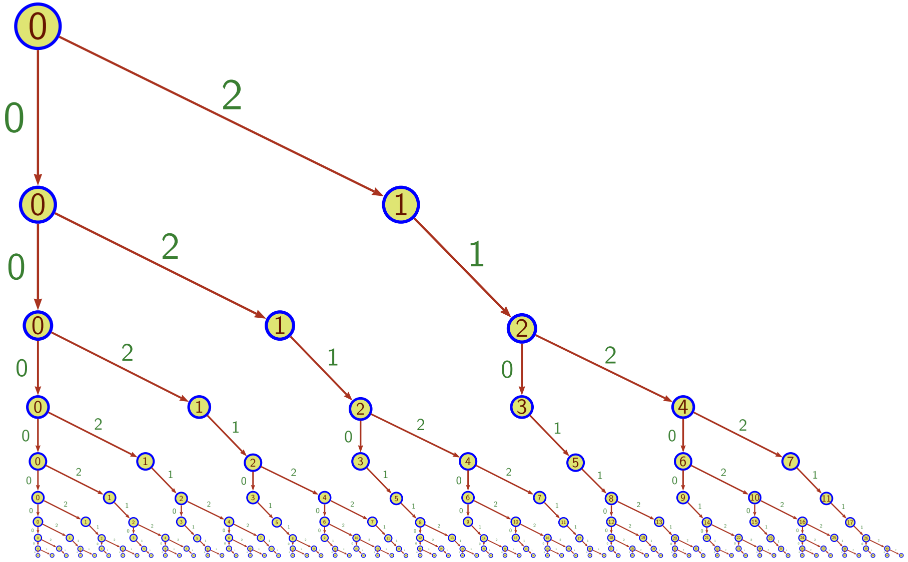
Every real in $[0,1]$ has at most two expansions in base 3 .

The set of reals in $[0,1]$ which have two expansions is infinite countable.

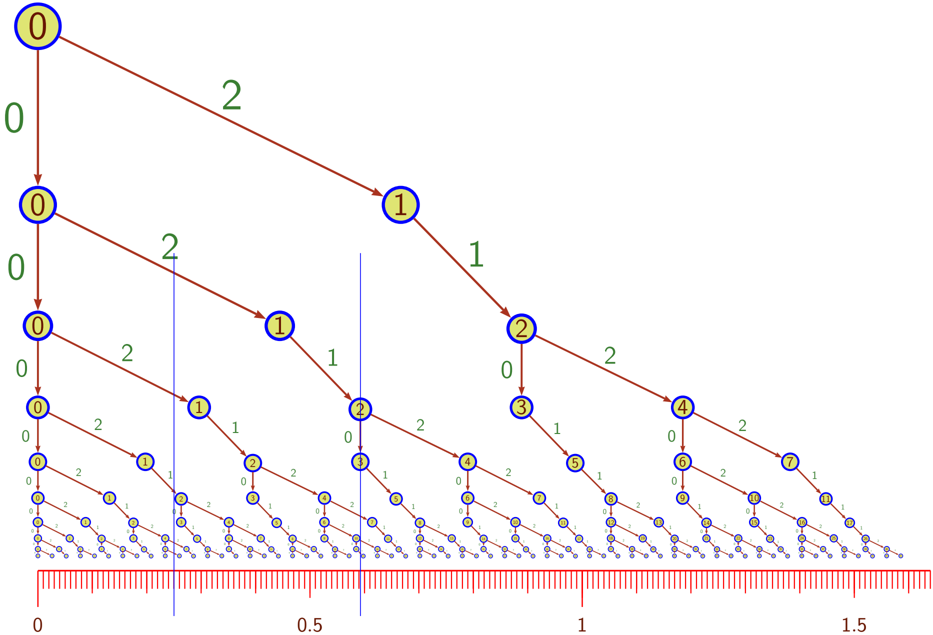
Representation of reals in base $\frac{3}{2}$: the tree $T'_{\frac{3}{2}}$



Representation of reals in base $\frac{3}{2}$: the tree $T'_{\frac{3}{2}}$



Representation of reals in base $\frac{3}{2}$: the tree $T'_{\frac{3}{2}}$



Representation of reals in base $\frac{3}{2}$: the tree $T'_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T'_{\frac{3}{2}}$ $\qquad$ $\mathbf{a} = \{a_i\}_{i \geqslant 1} \in A_3^{\mathbb{N}}$

Definition

$\mathbf{a}$ is an expansion in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

Such an $\mathbf{a}$ is a $\frac{3}{2}$-expansion of the real x defined by:

$$x = \pi(.\mathbf{a}) = \sum_{i \geqslant 1} a_i \, \frac{1}{2} \left(\frac{2}{3}\right)^i .$$

Representation of reals in base $\frac{3}{2}$: the tree $T'_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T'_{\frac{3}{2}}$ $\quad \mathbf{a} = \{a_i\}_{i \geqslant 1} \in A_3^{\mathbb{N}}$

Definition

$\mathbf{a}$ is an expansion in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

Such an $\mathbf{a}$ is a $\frac{3}{2}$-expansion of the real x defined by:

$$x = \pi(.\mathbf{a}) = \sum_{i \geqslant 1} a_i \, \frac{1}{2} \left(\frac{2}{3}\right)^i .$$

$W_{\frac{3}{2}}$ contains a maximal word. $\mathbf{t}_{\frac{3}{2}}$ $\quad \boldsymbol{\omega}_{\frac{3}{2}} = \pi(.\mathbf{t}_{\frac{3}{2}})$

$$\mathbf{t}_{\frac{3}{2}} = 212211122121122121211221 \cdots$$

$$\langle \boldsymbol{\omega}_{\frac{3}{2}} \rangle_{10} = 1.6222705028847673159569509 82 \cdots$$

Representation of reals in base $\frac{3}{2}$: the tree $T'_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T'_{\frac{3}{2}}$ $\qquad$ $\mathbf{a} = \{a_i\}_{i \geqslant 1} \in A_3^{\mathbb{N}}$

Definition

$\mathbf{a}$ is an expansion in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

Such an $\mathbf{a}$ is a $\frac{3}{2}$ -expansion of the real x defined by:

$$x = \pi(.\mathbf{a}) = \sum_{i \geqslant 1} a_i \frac{1}{2} \left(\frac{2}{3}\right)^i .$$

Representation of reals in base $\frac{3}{2}$: the tree $T'_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T'_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geqslant 1} \in A_3^{\mathbb{N}}$

Definition

$\mathbf{a}$ is an expansion in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

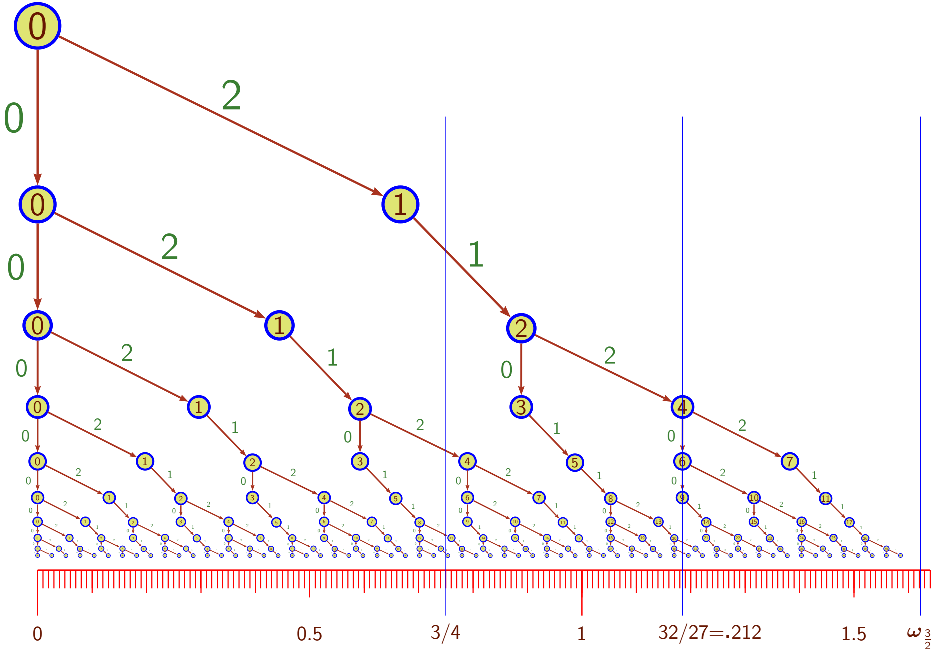
Such an $\mathbf{a}$ is a $\frac{3}{2}$ -expansion of the real x defined by:

$$x = \pi(.\mathbf{a}) = \sum_{i \geqslant 1} a_i \, \frac{1}{2} \left(\frac{2}{3}\right)^i .$$

Theorem (A.-F.-S. 05)

Every real of $[0, \boldsymbol{\omega}_{\frac{3}{2}}]$ *has (at least) one* $\frac{3}{2}$ *-expansion.*

Representation of reals in base $\frac{3}{2}$: the tree $T'_{\frac{3}{2}}$



Multiples $\frac{3}{2}$-expansions

The set of reals of $[0, \omega_{\frac{3}{2}}]$ that have more than one $\frac{3}{2}$-expansion is infinite countable.

Multiples $\frac{3}{2}$-expansions

The set of reals of $[0, \omega_{\frac{3}{2}}]$ that have more than one $\frac{3}{2}$-expansion is infinite countable.

No element of $W_{\frac{3}{2}}$, but 0^ω, is eventually periodic.

Multiples $\frac{3}{2}$-expansions

The set of reals of $[0, \omega_{\frac{3}{2}}]$ that have more than one $\frac{3}{2}$-expansion is infinite countable.

No element of $W_{\frac{3}{2}}$, but 0^ω, is eventually periodic.

The finite prefixes of a $\frac{3}{2}$-expansion, completed with 0's, are not $\frac{3}{2}$-expansions.

Multiples $\frac{3}{2}$-expansions

The set of reals of $[0, \omega_{\frac{3}{2}}]$ that have more than one $\frac{3}{2}$-expansion is infinite countable.

No element of $W_{\frac{3}{2}}$, but 0^{ω}, is eventually periodic.

The finite prefixes of a $\frac{3}{2}$-expansion, completed with 0's, are not $\frac{3}{2}$-expansions.

Every real in $[0, \omega_{\frac{3}{2}}]$ has at most two $\frac{3}{2}$-expansions.

Multiples $\frac{3}{2}$-expansions

The set of reals of $[0, \omega_{\frac{3}{2}}]$ that have more than one $\frac{3}{2}$-expansion is infinite countable.

No element of $W_{\frac{3}{2}}$, but 0^ω, is eventually periodic.

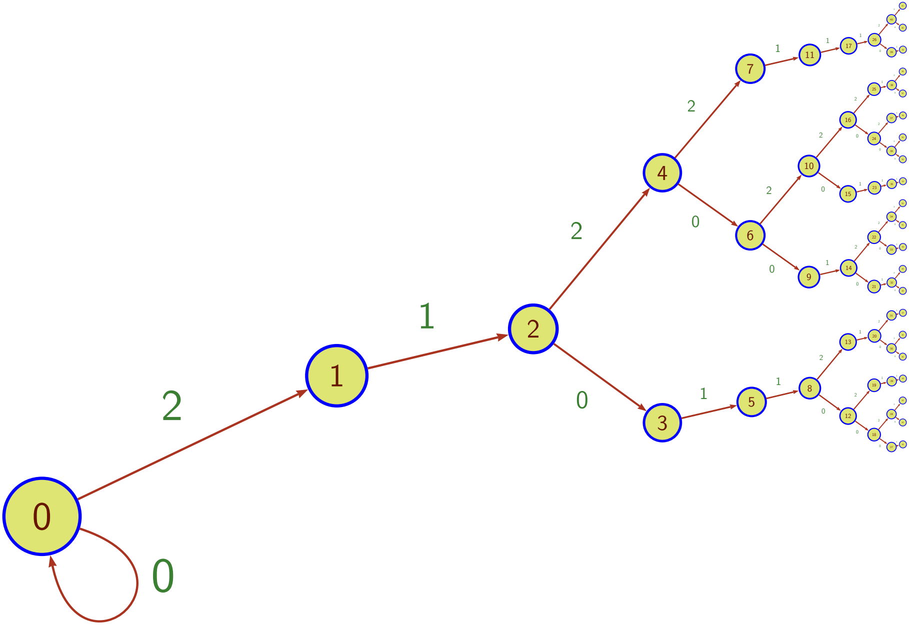
The finite prefixes of a $\frac{3}{2}$-expansion, completed with 0's, are not $\frac{3}{2}$-expansions.

Every real in $[0, \omega_{\frac{3}{2}}]$ has at most two $\frac{3}{2}$-expansions.

The set of reals in $[0, \omega_{\frac{3}{2}}]$ which have two expansions is infinite countable.

Part V

When order generates disorder



0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17

Meta theorem

The $T_{\frac{p}{q}}$ are characterised by their *periodic signature*.

Signature of a tree

Definition

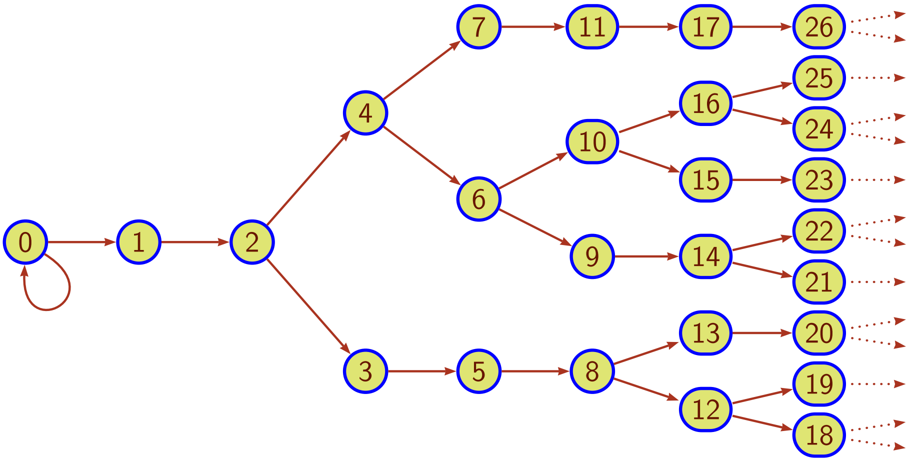
Signature of an ordered tree $\mathcal{T}$ =

sequence of the degrees of the nodes

in the breadth-first traversal of $\mathcal{T}$

Signature of a tree

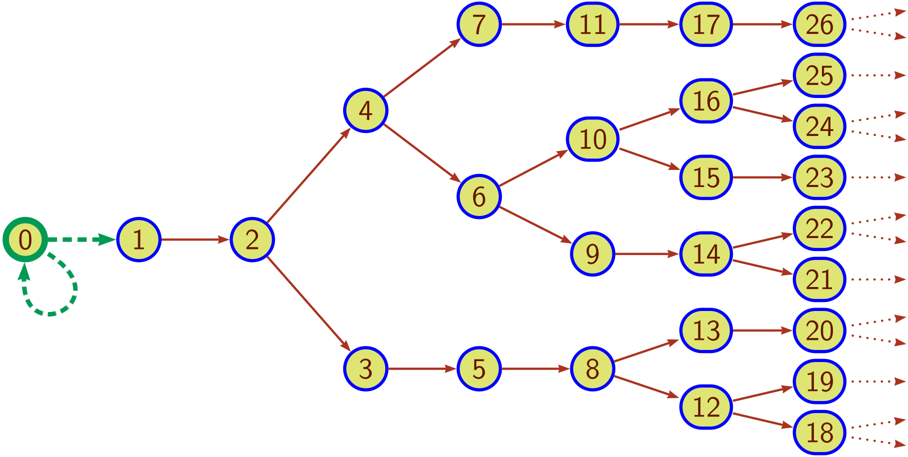
Signature = sequence of the degrees



s =

Signature of a tree

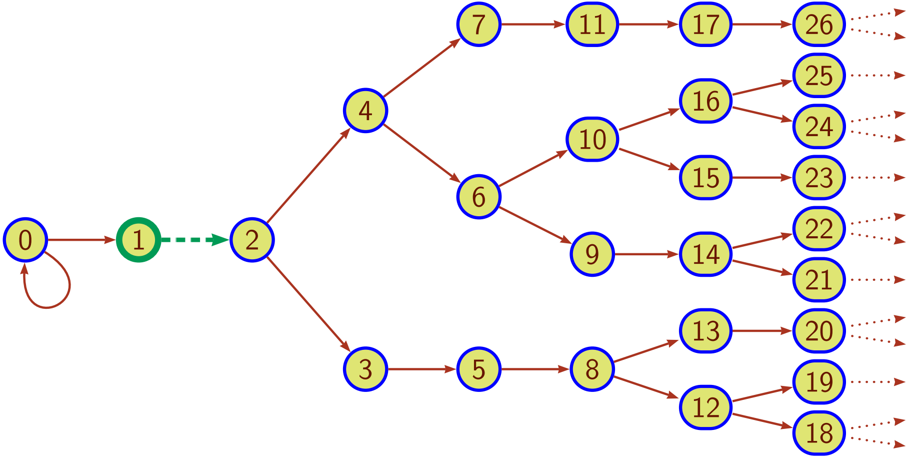
Signature = sequence of the degrees



$\mathbf{s} = 2$

Signature of a tree

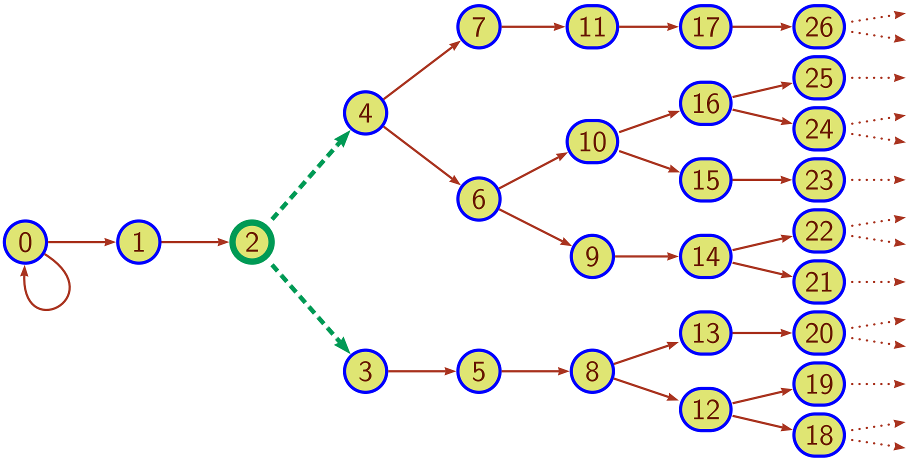
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1$

Signature of a tree

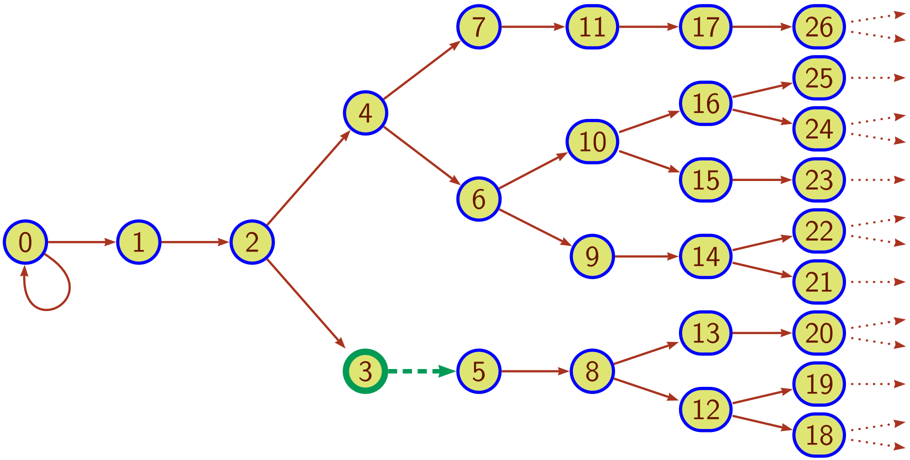
Signature = sequence of the degrees



$\mathbf{s} = \ 2 \ 1 \ 2$

Signature of a tree

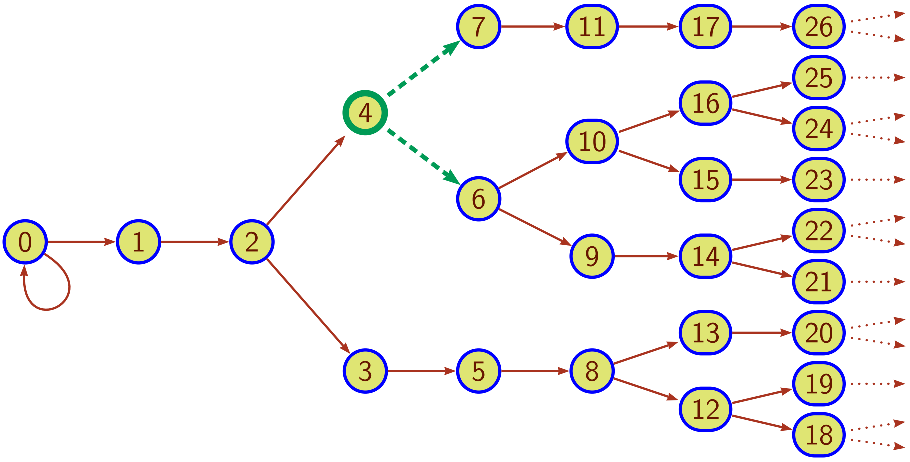
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1$

Signature of a tree

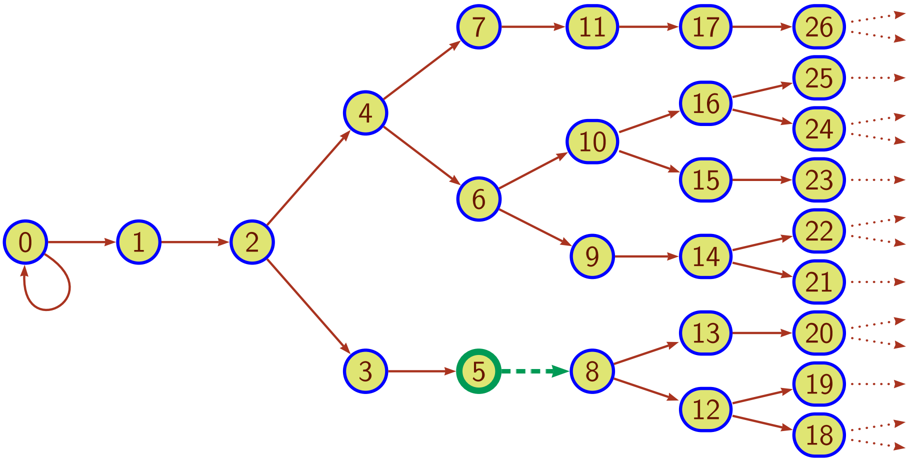
Signature = sequence of the degrees



s = 2 1 2 1 2

Signature of a tree

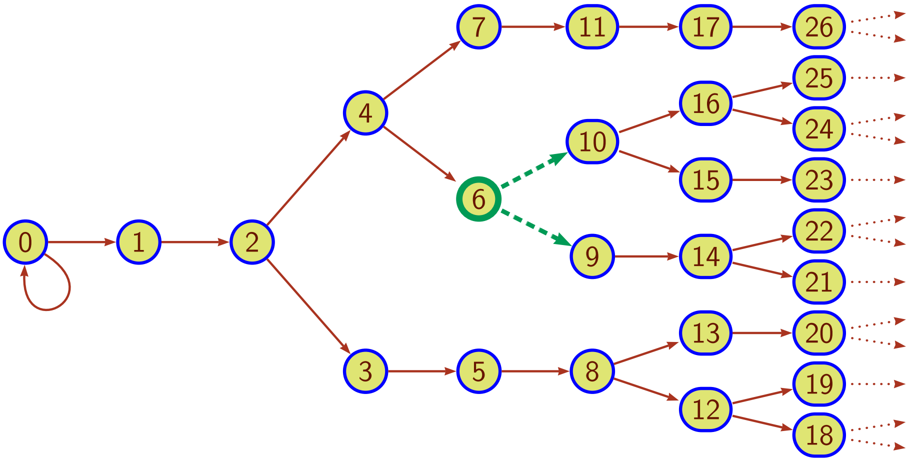
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1$

Signature of a tree

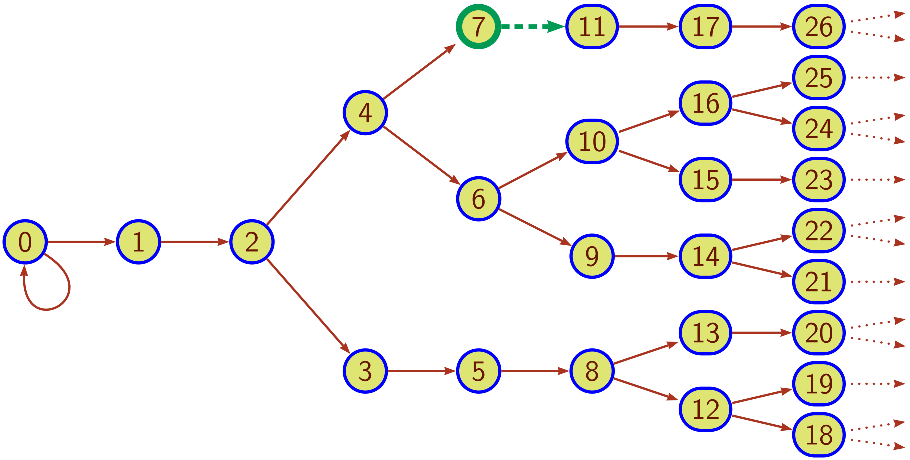
Signature = sequence of the degrees



$$\mathbf{s} = \; 2\ 1\ 2\ 1\ 2\ 1\ 2$$

Signature of a tree

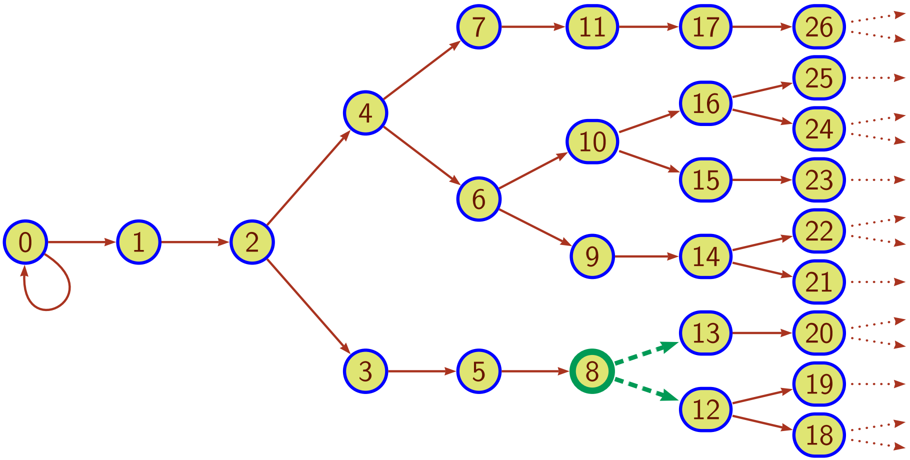
Signature = sequence of the degrees



$\mathbf{s} = \ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1$

Signature of a tree

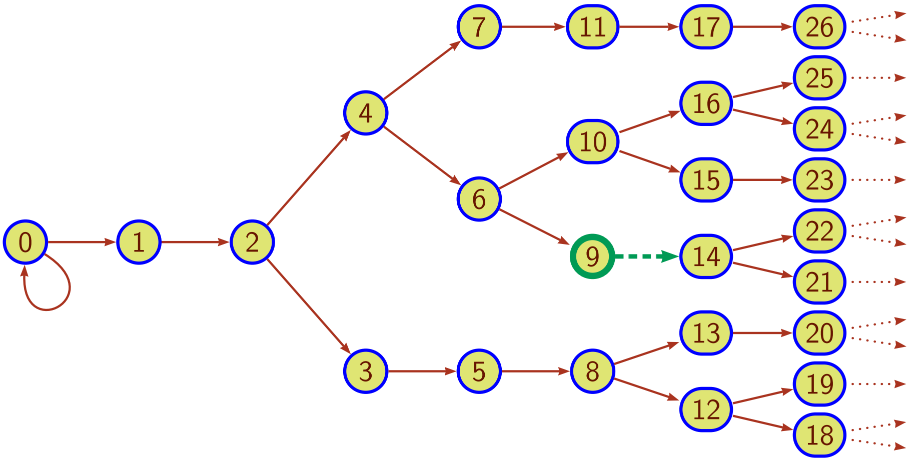
Signature = sequence of the degrees



$\mathbf{s} =\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2$

Signature of a tree

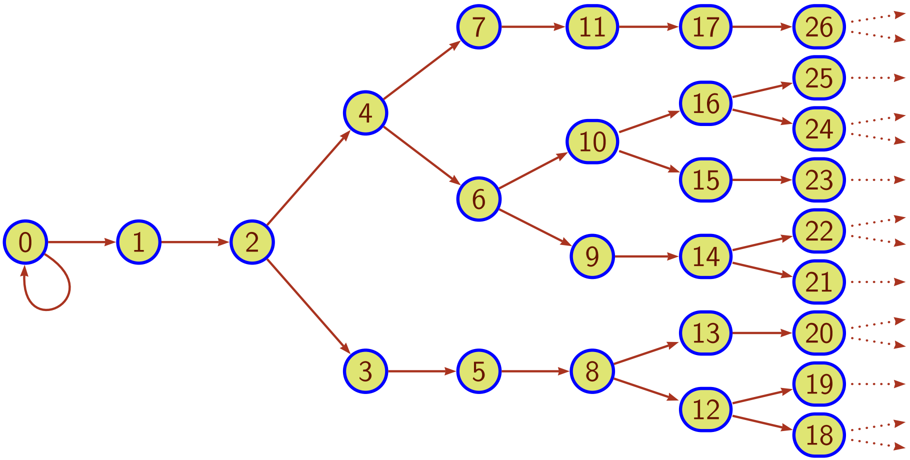
Signature = sequence of the degrees



$\mathbf{s} = \ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1$

Signature of a tree

Signature = sequence of the degrees



$$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

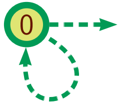
Tree from a signature

Signature = sequence of the degrees

$$\mathbf{s} = \; 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Tree from a signature

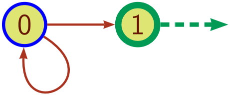
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

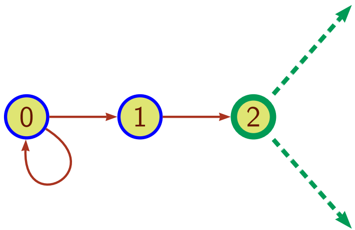
Signature = sequence of the degrees



$$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Tree from a signature

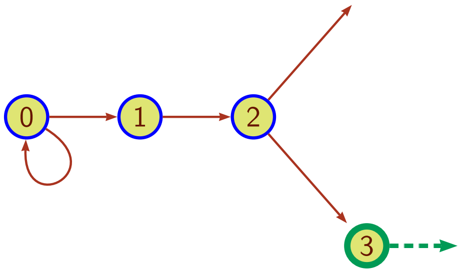
Signature = sequence of the degrees



$\mathbf{s} = \ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

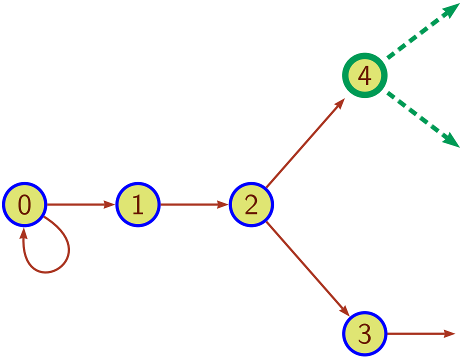
Signature = sequence of the degrees



$\mathbf{s} = \ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

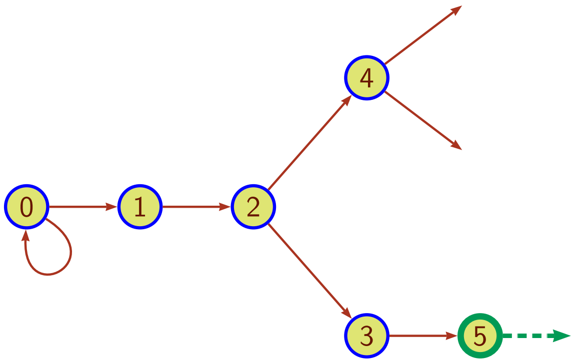
Signature = sequence of the degrees



$\mathbf{s} = \; 2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;\cdots$

Tree from a signature

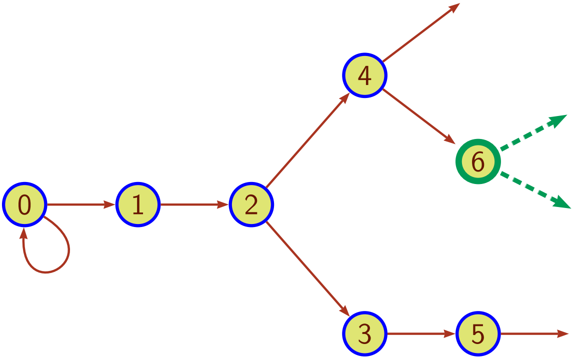
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

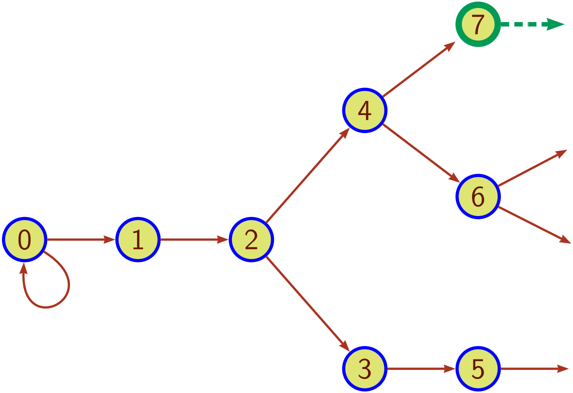
Signature = sequence of the degrees



$\mathbf{s} = \; 2\;1\;2\;1\;2\;1\;\textcolor{red}{2}\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;\cdots$

Tree from a signature

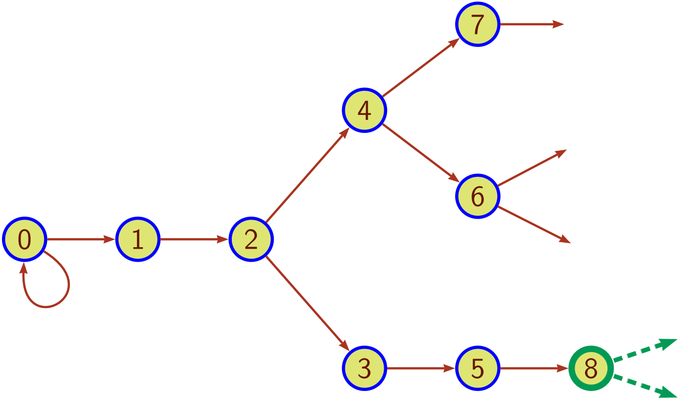
Signature = sequence of the degrees



$$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Tree from a signature

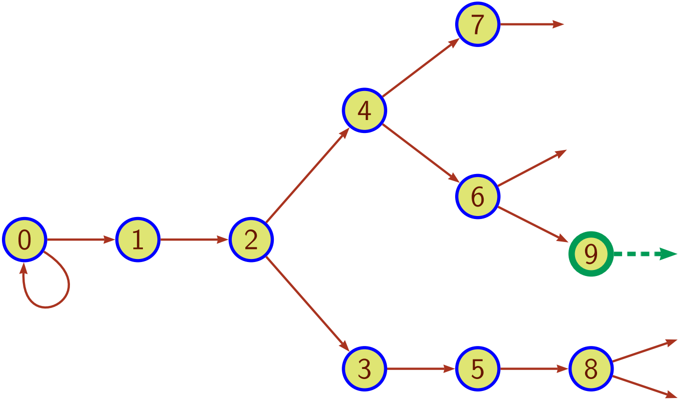
Signature = sequence of the degrees



$\mathbf{s} = \ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

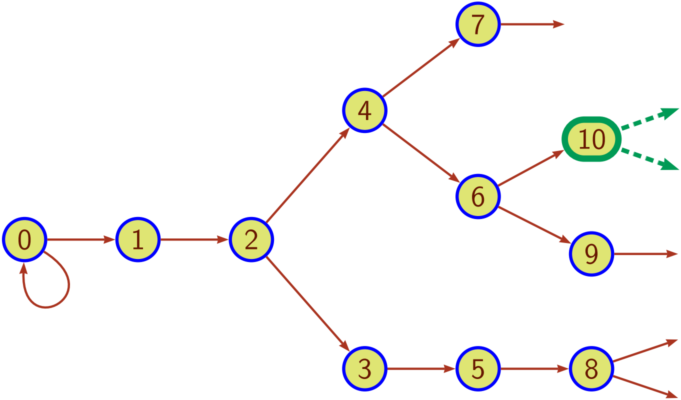
Signature = sequence of the degrees



$\mathbf{s} = \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$

Tree from a signature

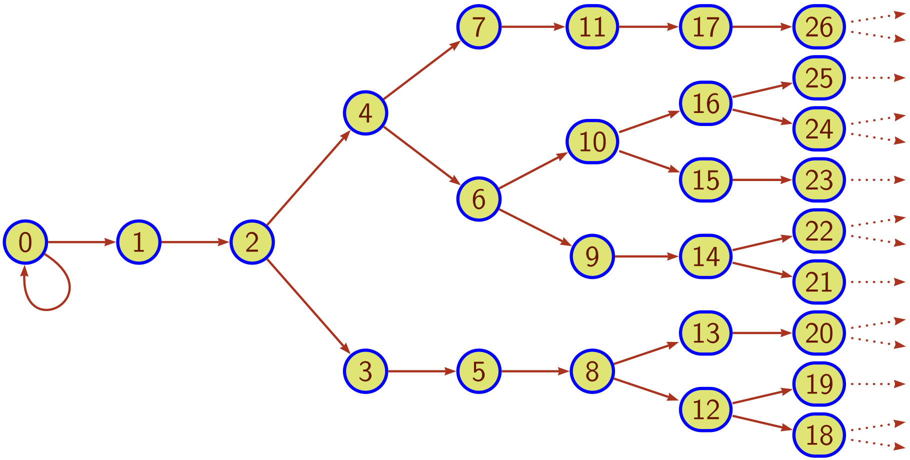
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

Signature = sequence of the degrees



$\mathbf{s} = \; 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Labelled signature of a labelled tree

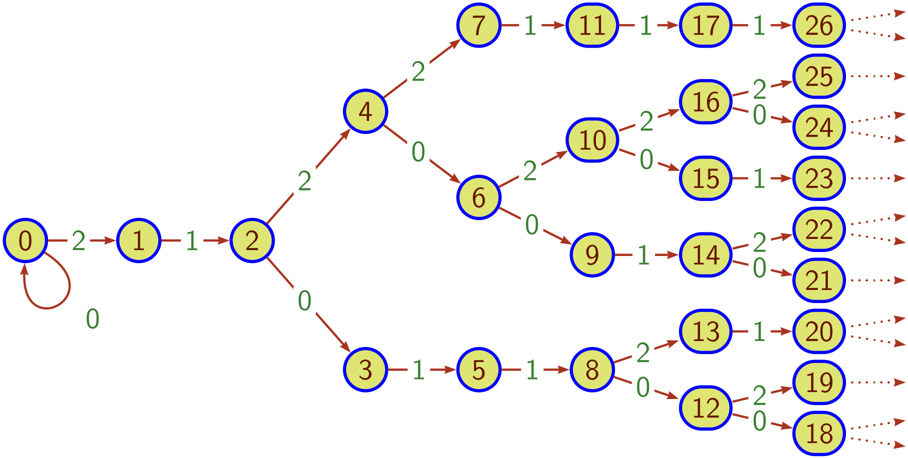
Arcs of $\mathcal{T}$ labelled in an ordered alphabet A

Definition

Labelled signature of an ordered tree $\mathcal{T}$ =
signature of $\mathcal{T}$ +
sequence of the labels of the arcs
in the breadth-first traversal of $\mathcal{T}$

labelled signature $(\mathbf{s}, \boldsymbol{\lambda})$

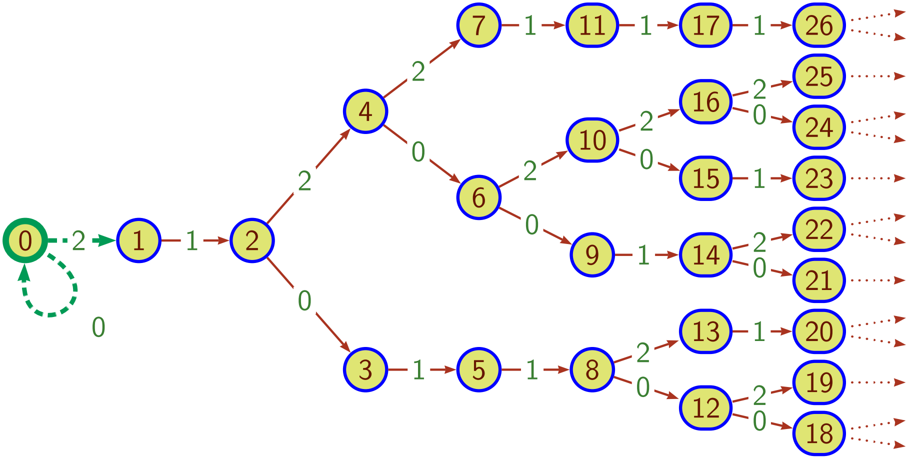
Labelled signature of a labelled tree



$\mathbf{s} =$

$\boldsymbol{\lambda} =$

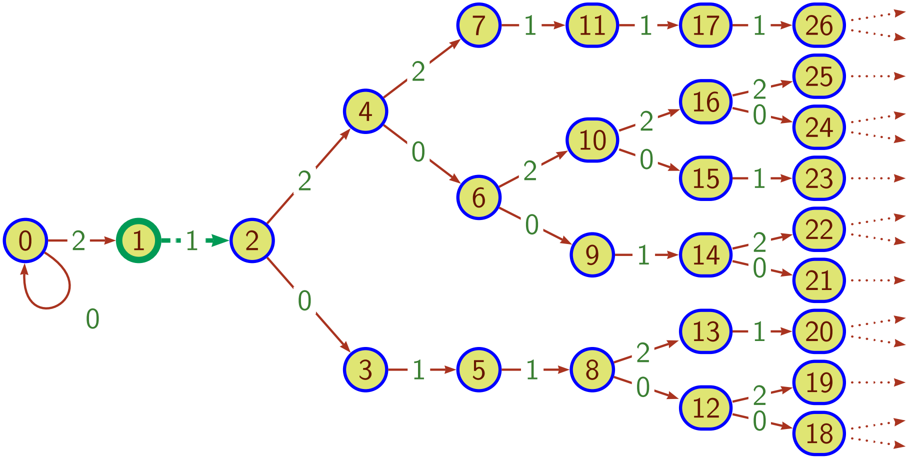
Labelled signature of a labelled tree



$\mathbf{s} = 2$

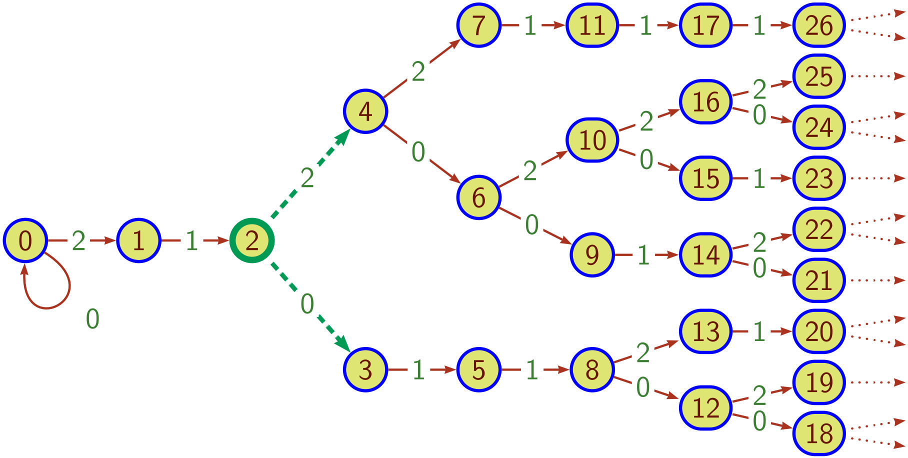
$\boldsymbol{\lambda} = 0\,2$

Labelled signature of a labelled tree



$$\mathbf{s} = \;\; 2 \;\; 1$$
$$\boldsymbol{\lambda} = 0\,2\,1$$

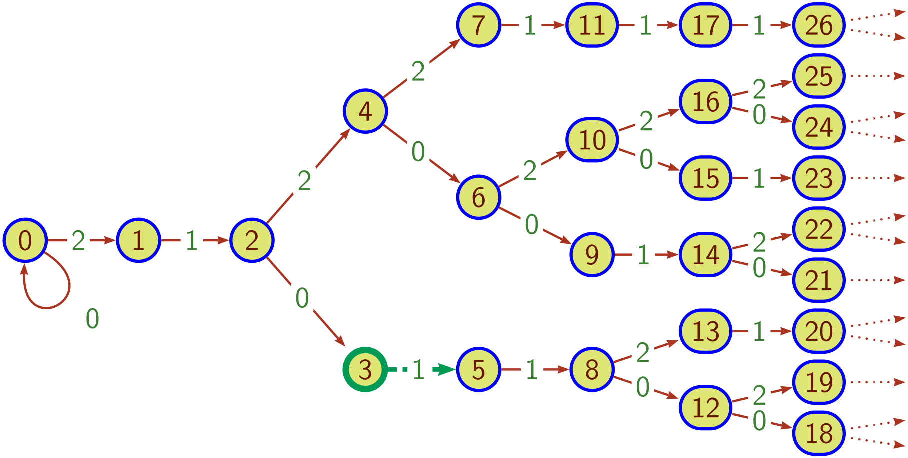
Labelled signature of a labelled tree



$$\mathbf{s} = \ 2\ 1\ 2$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2$$

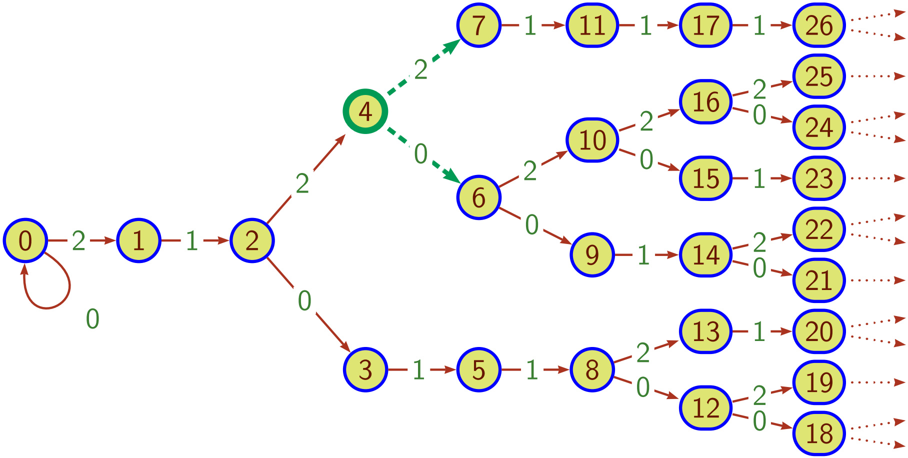
Labelled signature of a labelled tree



$\mathbf{s} = \ 2\ 1\ 2\ 1$

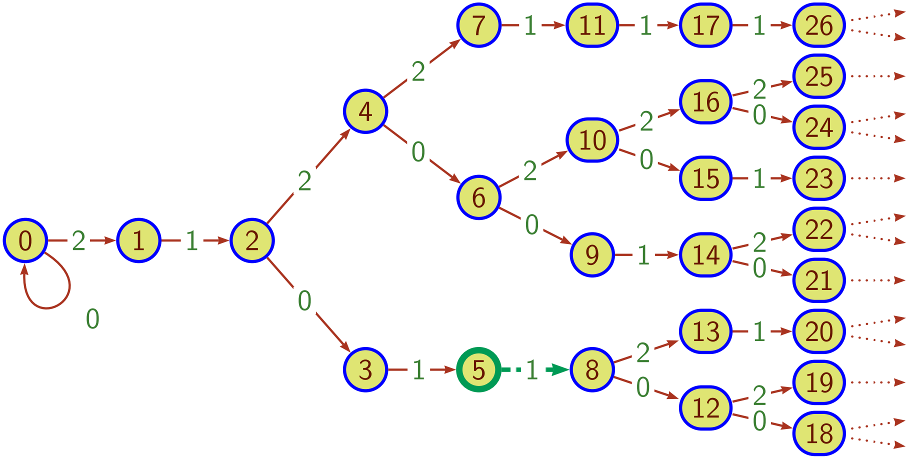
$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1$

Labelled signature of a labelled tree



$$\mathbf{s} = \; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2$$
$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2$$

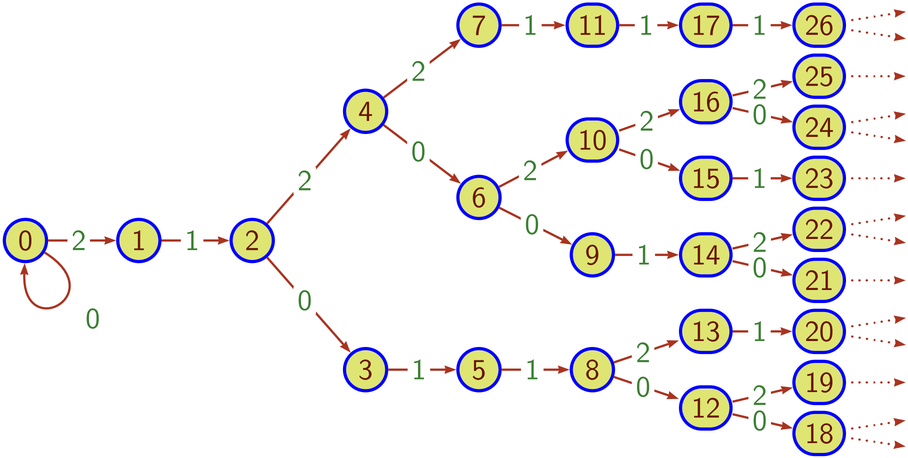
Labelled signature of a labelled tree



s = 2 1 2 1 2 1

λ = 0 2 1 0 2 1 0 2 1

Labelled signature of a labelled tree



$$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

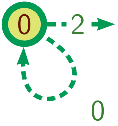
$$\boldsymbol{\lambda} = 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ \cdots$$

Labelled tree from a labelled signature

$$\mathbf{s} = \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,\cdots$$

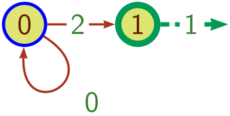
Labelled tree from a labelled signature



$$\mathbf{s} = \ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

$$\boldsymbol{\lambda} = 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ \cdots$$

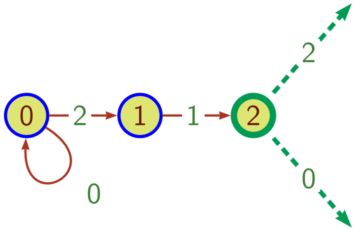
Labelled tree from a labelled signature



$\mathbf{s} = \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$

$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1 \; \cdots$

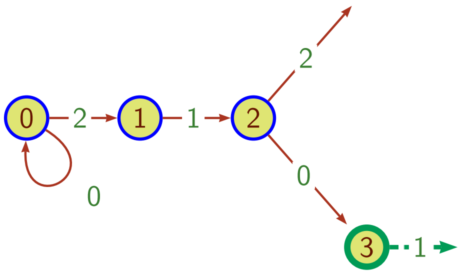
Labelled tree from a labelled signature



$$\mathbf{s} = \; 2 \; 1 \; {\color{red}2} \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\,{\color{red}0\,2}\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1 \; \cdots$$

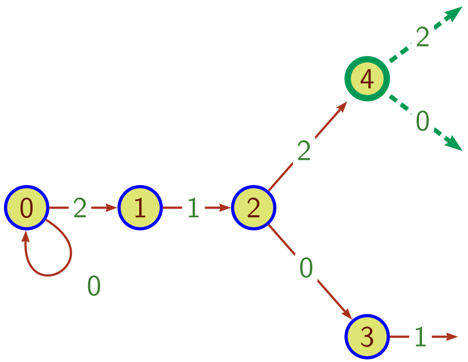
Labelled tree from a labelled signature



$$\mathbf{s} = \; 2 \; 1 \; 2 \; {\color{red}1} \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,{\color{red}1}\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1 \; \cdots$$

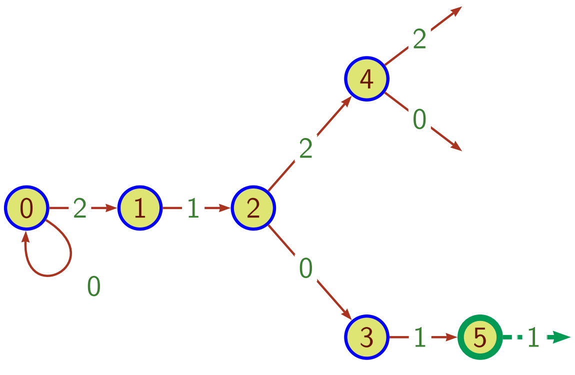
Labelled tree from a labelled signature



$$\mathbf{s} = \; 2 \; 1 \; 2 \; 1 \; \color{red}{2} \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$$

$$\boldsymbol{\lambda} = 0\,2 \; 1 \; 0\,2 \; 1 \; \color{red}{0\,2} \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; \cdots$$

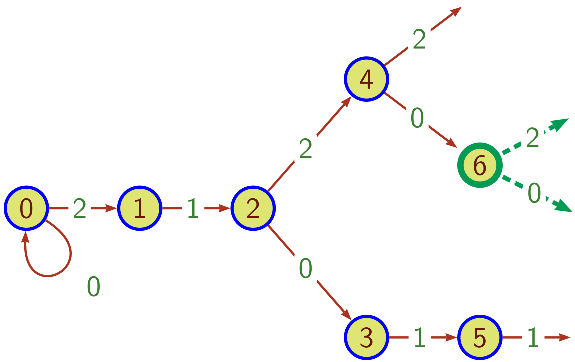
Labelled tree from a labelled signature



$$\mathbf{s} = \; 2\;1\;2\;1\;2\;{\color{red}1}\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;\cdots$$

$$\boldsymbol{\lambda} = 0\,2\;1\;0\,2\;1\;0\,2\;{\color{red}1}\;0\,2\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;\cdots$$

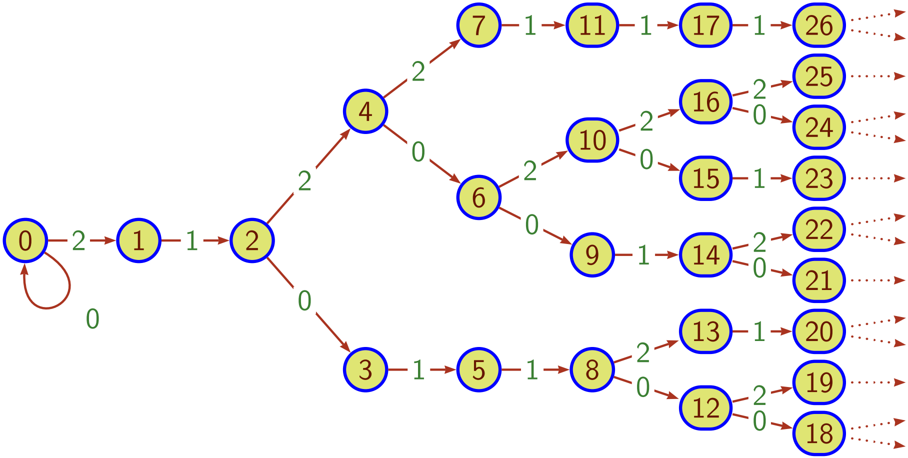
Labelled tree from a labelled signature



$$\mathbf{s} = \; 2\;1\;2\;1\;2\;1\;{\color{red}2}\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;\cdots$$

$$\boldsymbol{\lambda} = 0\,2\;1\;0\,2\;1\;0\,2\;1\;{\color{red}0\,2}\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;\cdots$$

Labelled tree from a labelled signature



$$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ 0\,2\,1\ \cdots$$

Signature of $T_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

Signature of $T_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

Theorem

The (labelled) signature of $T_{\frac{p}{q}}$ *is purely periodic.*

Rhythm

p, q coprime integers $p > q \geqslant 1$

A purely periodic signature

$$\mathbf{s} = \mathbf{r}^{\omega}$$

Rhythm

p, q coprime integers $p > q \geqslant 1$

A purely periodic signature

$$\mathbf{s} \;=\; \mathbf{r}^{\omega}$$

Definition

$\mathbf{r}$ rhythm of directing parameter (q, p)

$$\mathbf{r} \;=\; (r_0, r_1, \ldots, r_{q-1}) \qquad\qquad \sum_{i=0}^{q-1} r_i \;=\; p$$

Rhythm

p, q coprime integers $p > q \geqslant 1$

A purely periodic signature

$$\mathbf{s} \ = \ \mathbf{r}^{\omega}$$

Definition

$\mathbf{r}$ rhythm of directing parameter (q, p)

$$\mathbf{r} \ = \ (r_0, r_1, \ldots, r_{q-1}) \qquad\qquad \sum_{i=0}^{q-1} r_i \ = \ p$$

Example

Rhythms of dir. par. $(3, 5)$: $(3, 1, 1)$ $(2, 2, 1)$ $(1, 2, 2)$

Rhythm

p, q coprime integers $\; p > q \geqslant 1$

Geometric representation

$\mathbf{r} = (r_0, r_1, \dots, r_{q-1})$

$$\texttt{path}(\mathbf{r}) = y^{r_0} x \, y^{r_1} x \, y^{r_2} \cdots x \, y^{r_{q-1}} x$$

Rhythm

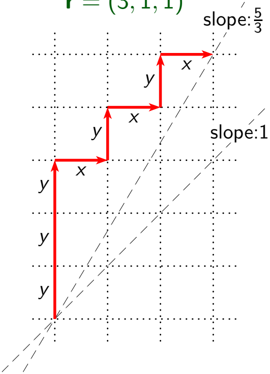
p, q coprime integers $p > q \geqslant 1$

Geometric representation

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$

$$\mathtt{path}(\mathbf{r}) = y^{r_0} x\, y^{r_1} x\, y^{r_2} \cdots x\, y^{r_{q-1}} x$$

$\mathbf{r} = (3, 1, 1)$



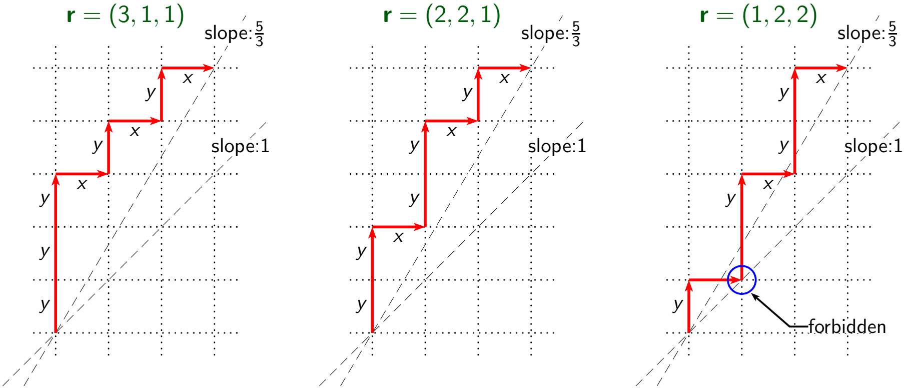
Rhythm

p, q coprime integers $p > q \geqslant 1$

Geometric representation

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$

$$\texttt{path}(\mathbf{r}) = y^{r_0} x\, y^{r_1} x\, y^{r_2} \cdots x\, y^{r_{q-1}} x$$



Christoffel rhythm $\mathbf{r}_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ Christoffel rhythm if $\texttt{path}(\mathbf{r})$ Christoffel word

Christoffel rhythm $\mathbf{r}_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ Christoffel rhythm if $\texttt{path}(\mathbf{r})$ Christoffel word

$\texttt{path}(\mathbf{r})$ Christoffel word if no integer point between $\texttt{path}(\mathbf{r})$ and slope

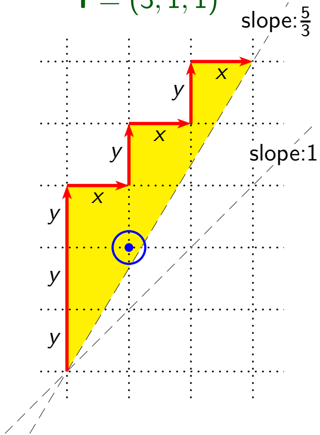
Christoffel rhythm $\mathbf{r}_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

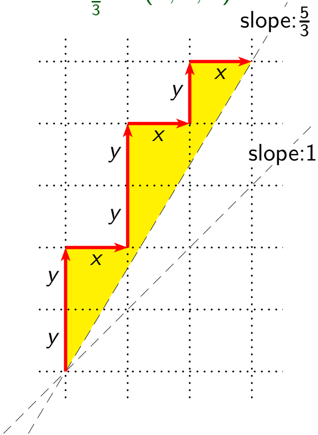
$\mathbf{r}$ Christoffel rhythm if $\mathtt{path}(\mathbf{r})$ Christoffel word

$\mathtt{path}(\mathbf{r})$ Christoffel word if no integer point between $\mathtt{path}(\mathbf{r})$ and slope

$\mathbf{r} = (3,1,1)$



$\mathbf{r}_{\frac{5}{3}} = (2,2,1)$



Signature of $T_{\frac{p}{q}}$

p, q coprime integers, $p > q \geqslant 1$

Theorem

The signature of $T_{\frac{p}{q}}$ is purely periodic of period $\mathbf{r}_{\frac{p}{q}}$.

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} \; = \; \big(0, \, (q \,\%\, p), \, (2\,q \,\%\, p), \ldots, \, ((p-1)\,q \,\%\, p) \big) \; .$$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} = \big(0, (q \,\%\, p), (2q \,\%\, p), \ldots, ((p-1)\, q \,\%\, p)\big) .$$

Examples

$\mathbf{r}_{\frac{3}{2}} = (2, 1)$ $\gamma_{\frac{3}{2}} = 0\,2\,1$ $\mathbf{r}_{\frac{5}{3}} = (2, 2, 1)$ $\gamma_{\frac{5}{3}} = 0\,3\,1\,4\,2$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} = \big(0,\, (q \,\%\, p),\, (2\,q \,\%\, p), \ldots,\, ((p-1)\,q \,\%\, p)\big) \ .$$

Examples

$\mathbf{r}_{\frac{3}{2}} = (2, 1)$ $\gamma_{\frac{3}{2}} = 0\,2\,1$ $\mathbf{r}_{\frac{5}{3}} = (2, 2, 1)$ $\gamma_{\frac{5}{3}} = 0\,3\,1\,4\,2$

Proposition

$\gamma_{\frac{p}{q}}$ *is consistent with* $\mathbf{r}_{\frac{p}{q}}$

Signature of $T_{\frac{p}{q}}$

p, q coprime integers, $p > q \geqslant 1$

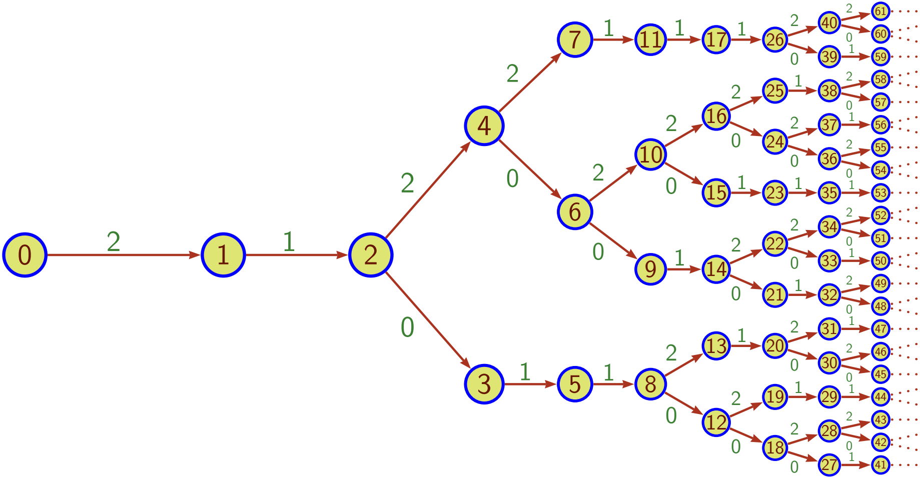
Theorem

The labelled signature of $T_{\frac{p}{q}}$ *is purely periodic of period* $(\mathbf{r}_{\frac{p}{q}}, \gamma_{\frac{p}{q}})$.

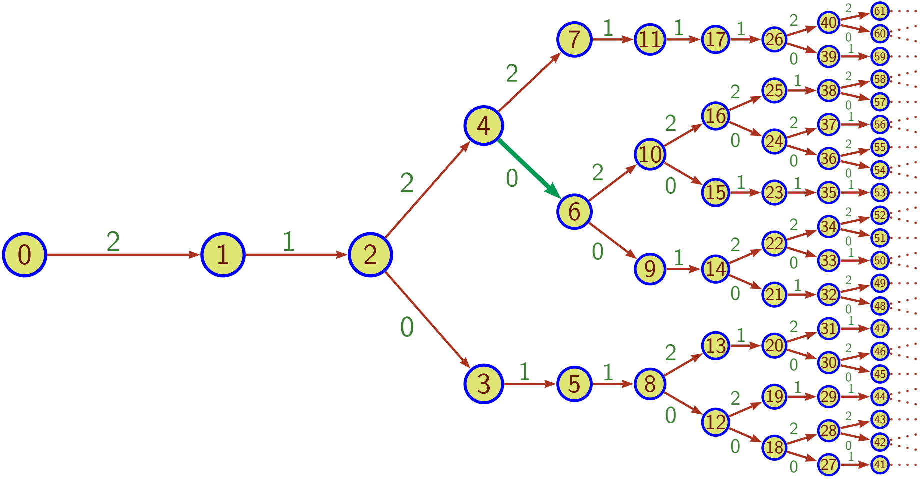
Part VI

A property still missing a proper name

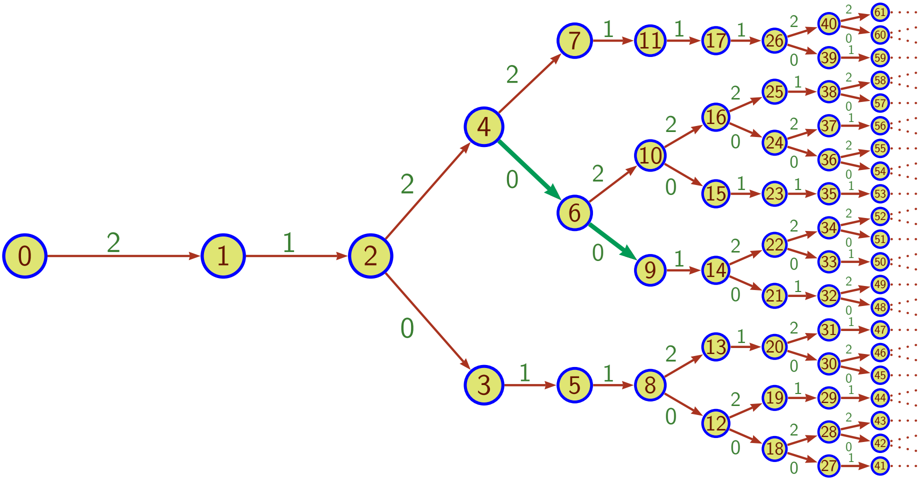
Minimal words in $T_{\frac{3}{2}}$



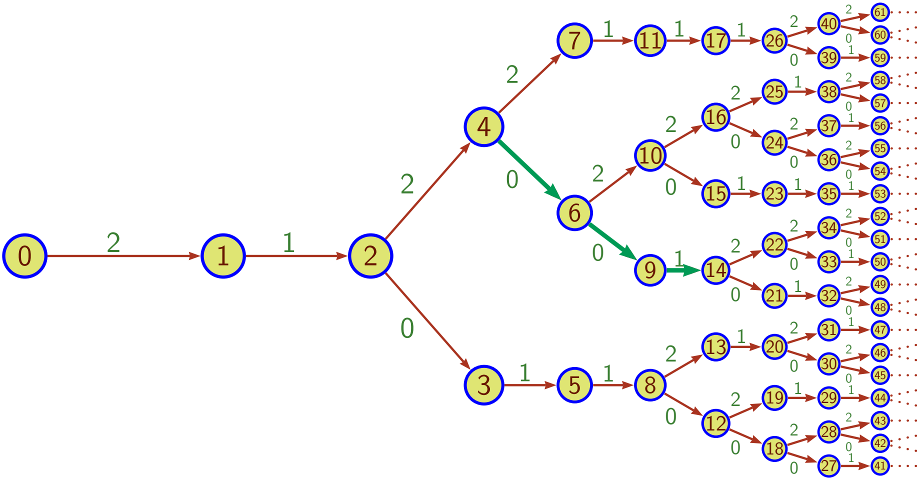
Minimal words in $T_{\frac{3}{2}}$



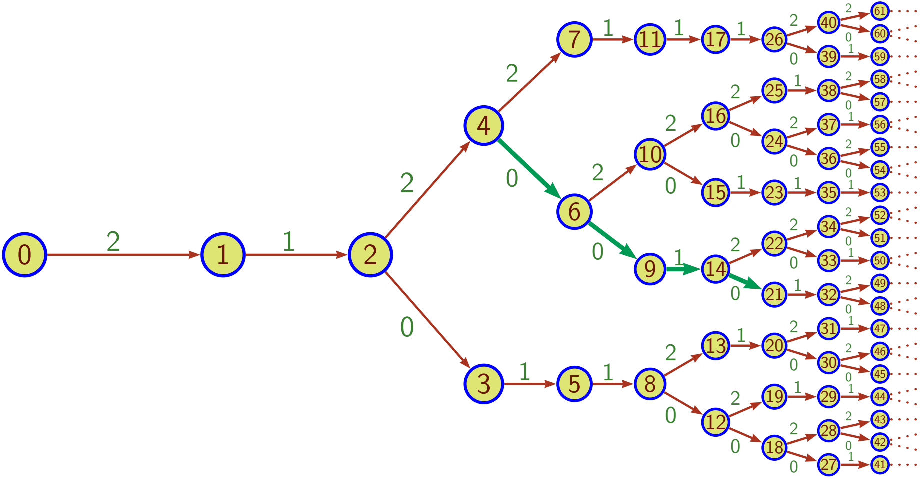
Minimal words in $T_{\frac{3}{2}}$



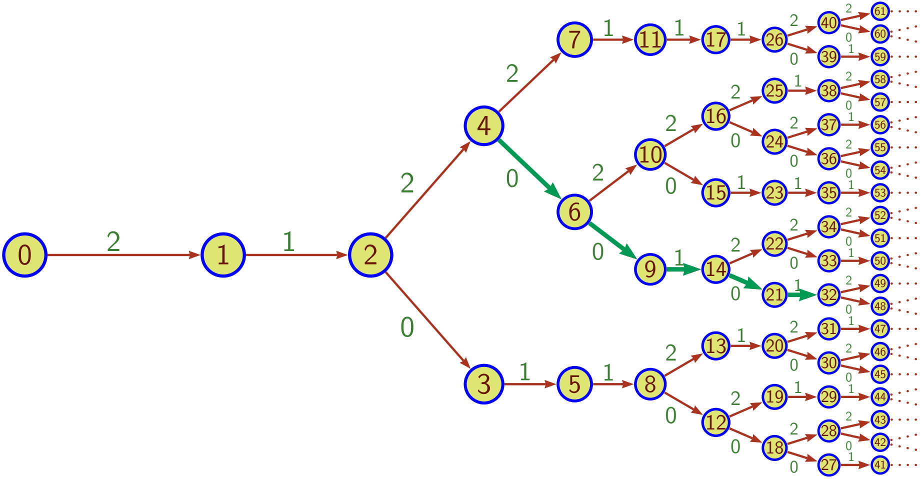
Minimal words in $T_{\frac{3}{2}}$



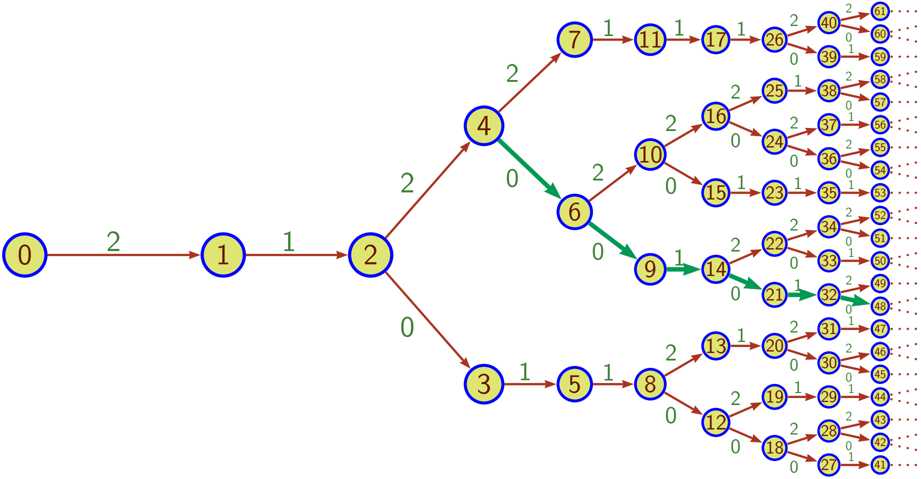
Minimal words in $T_{\frac{3}{2}}$



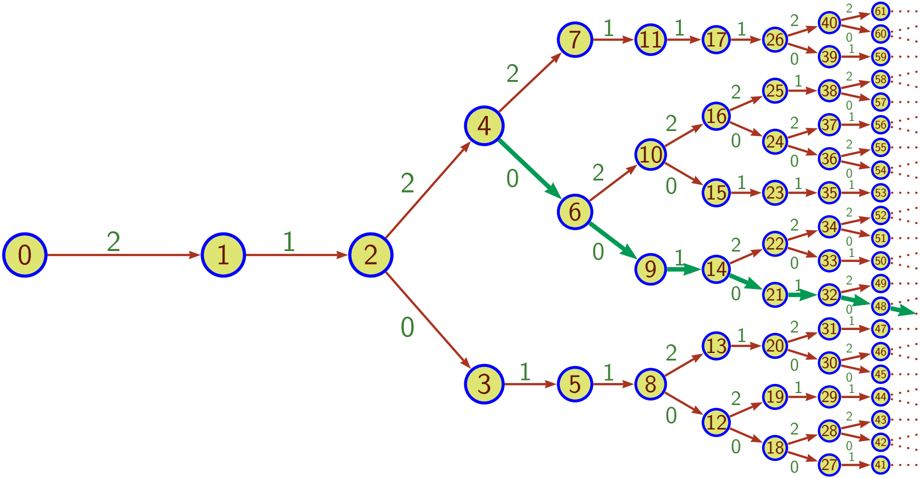
Minimal words in $T_{\frac{3}{2}}$



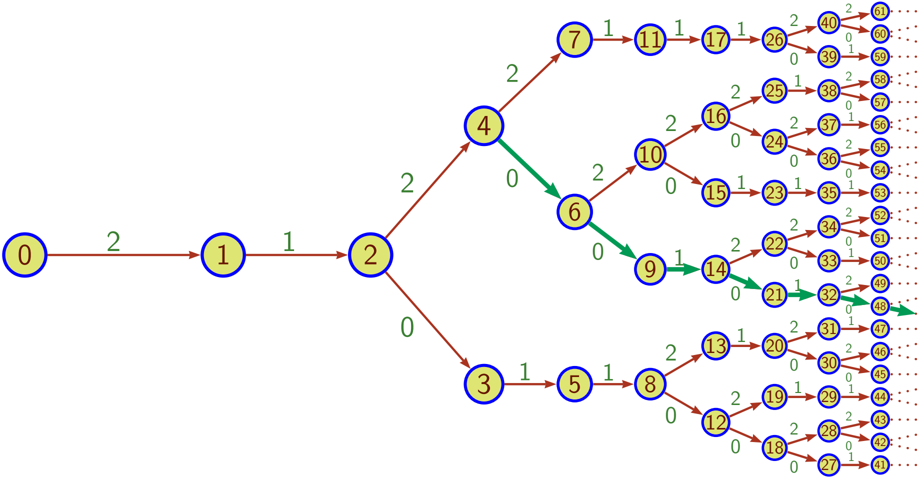
Minimal words in $T_{\frac{3}{2}}$



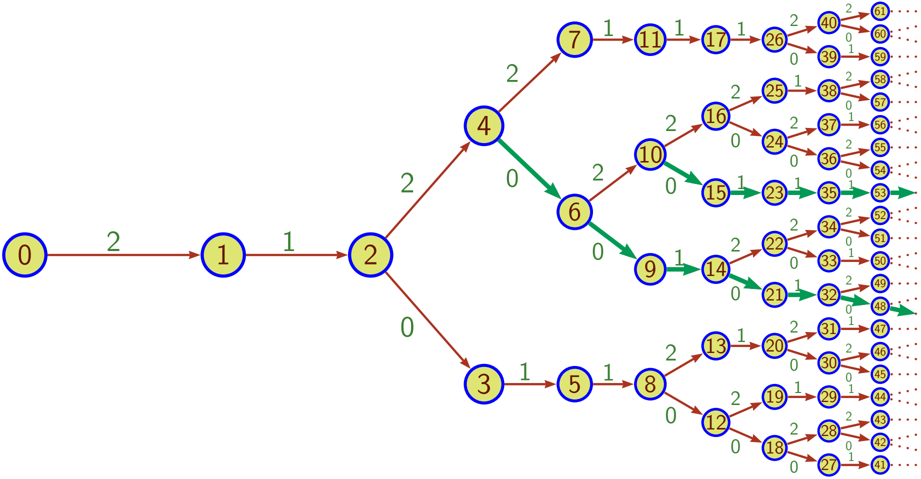
Minimal words in $T_{\frac{3}{2}}$



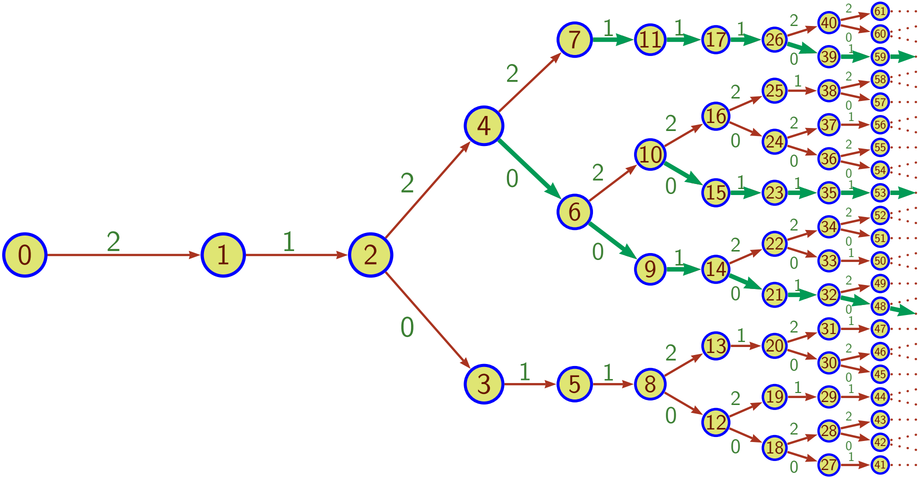
Minimal words in $T_{\frac{3}{2}}$



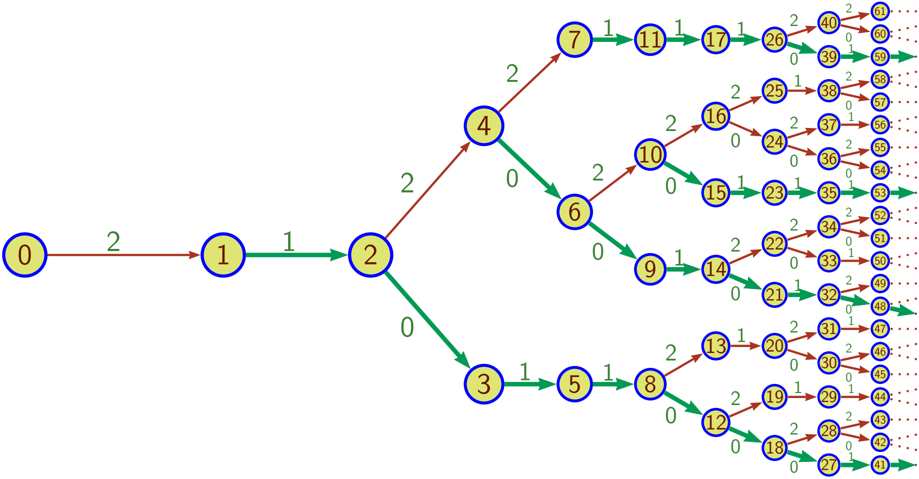
Minimal words in $T_{\frac{3}{2}}$



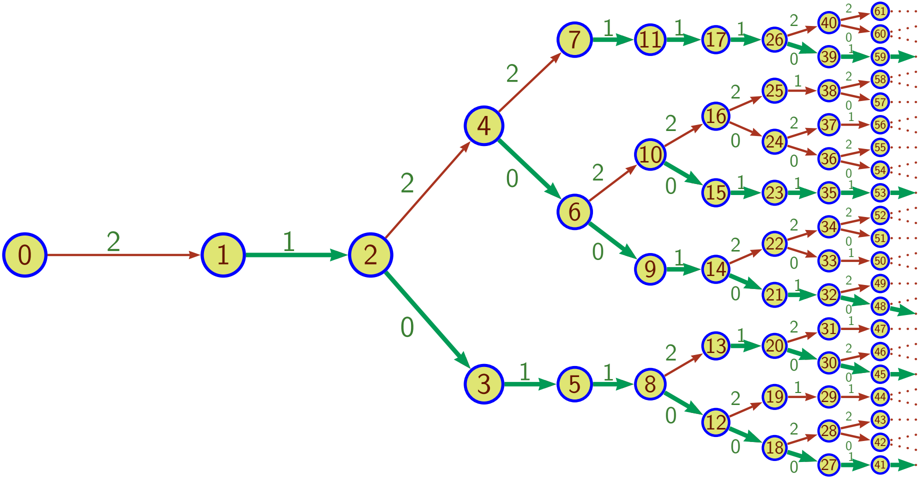
Minimal words in $T_{\frac{3}{2}}$



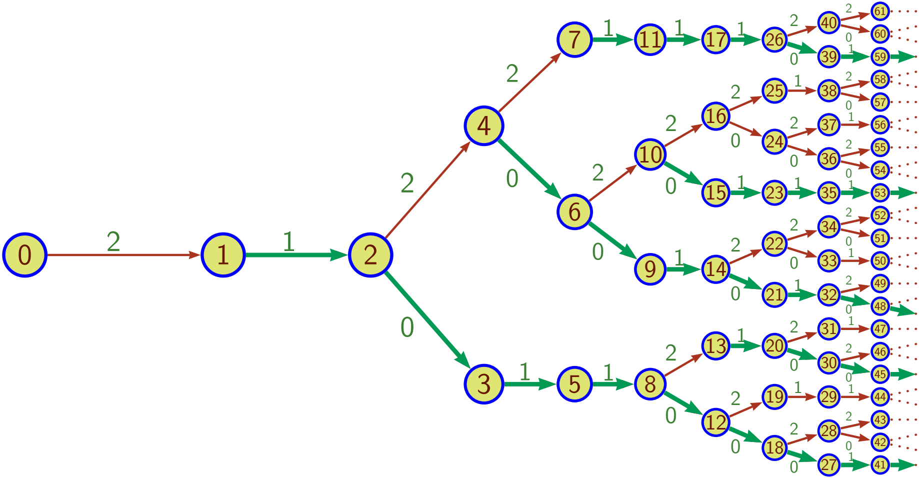
Minimal words in $T_{\frac{3}{2}}$



Minimal words in $T_{\frac{3}{2}}$



Minimal words in $T_{\frac{3}{2}}$



The $\mathbf{w}_n^-$ are all distinct words of $\{0,1\}^\omega$.

Minimal words in $T_{\frac{3}{2}}$

Problem

What is the relation between the $\mathbf{w}_n^-$?

Minimal words in $T_{\frac{3}{2}}$

Problem

What is the relation between the $\mathbf{w}_n^-$?

Conjecture?

For every n there exists a finite transducer that takes $\mathbf{w}_n^-$ as input and outputs $\mathbf{w}_{n+1}^-$.

Minimal words in $T_{\frac{3}{2}}$

Problem

What is the relation between the $\mathbf{w}_n^-$?

Conjecture?

For every n there exists a finite transducer
that takes $\mathbf{w}_n^-$ as input and outputs $\mathbf{w}_{n+1}^-$.

Stupid remark

True for $n = 1, 2$.

Minimal words in $T_{\frac{3}{2}}$

Problem

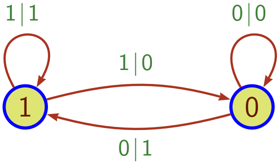
What is the relation between the $\mathbf{w}_n^-$?

Conjecture?

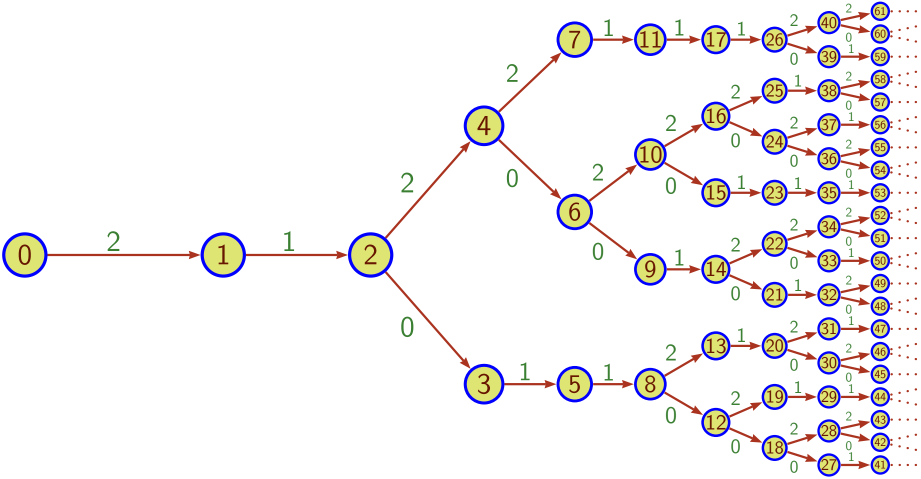
For every n there exists a finite transducer that takes $\mathbf{w}_n^-$ as input and outputs $\mathbf{w}_{n+1}^-$.

Stupid remark

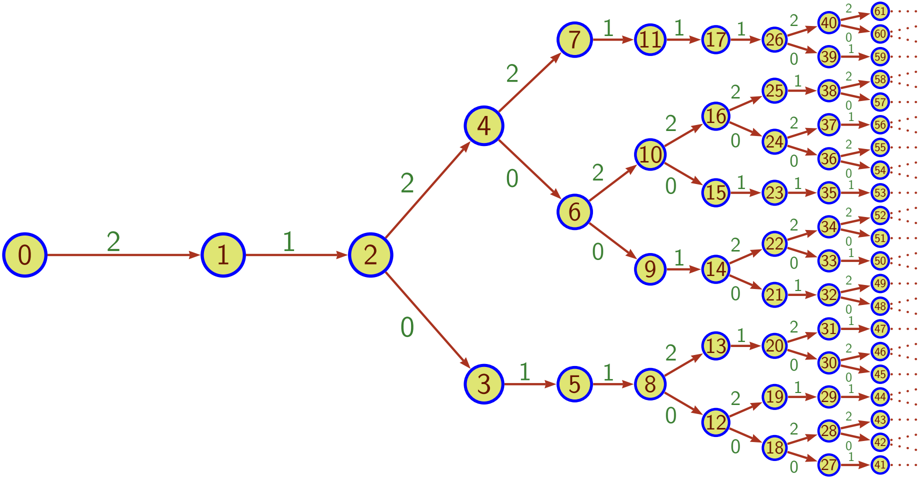
True for $n = 1, 2$.



Derived transducer $\mathcal{D}_{\frac{3}{2}}$

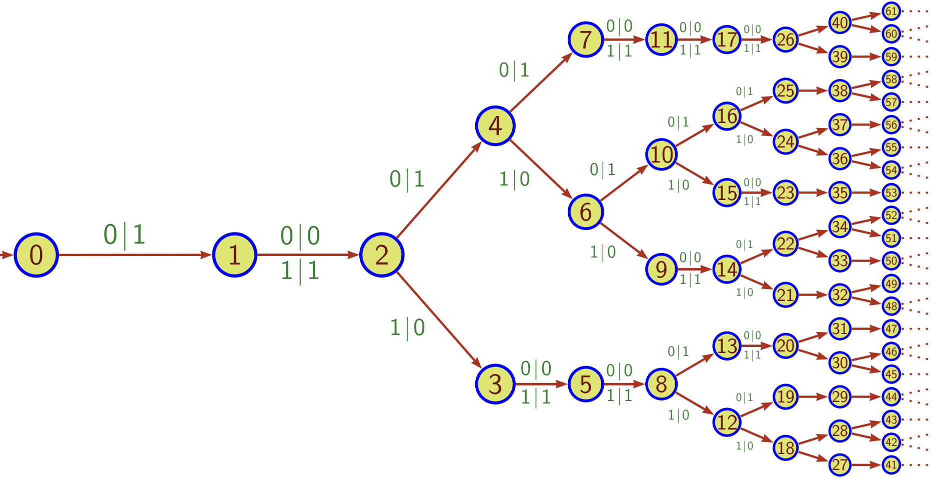


Derived transducer $\mathcal{D}_{\frac{3}{2}}$



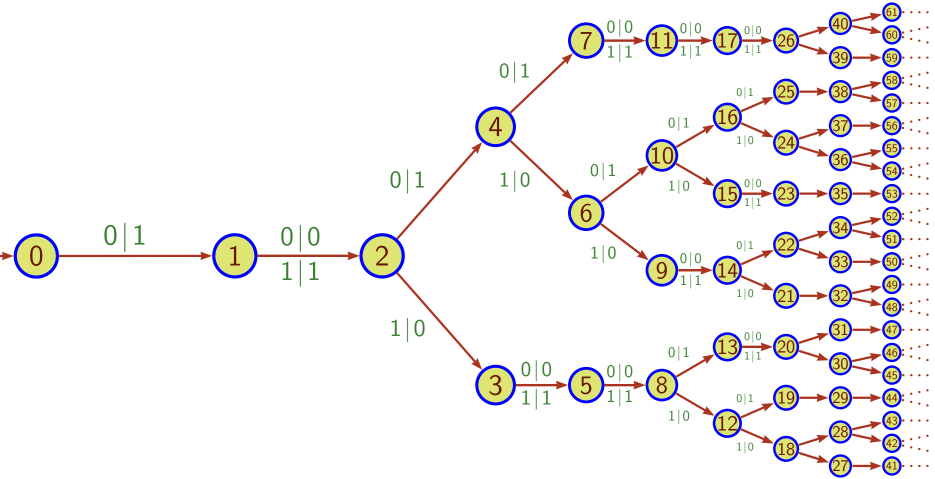
$2 \to 0\,|\,1 \qquad 1 \to 0\,|\,0,1\,|\,1 \qquad 0 \to 1\,|\,0$

Derived transducer $\mathcal{D}_{\frac{3}{2}}$



$2 \to 0|1 \qquad 1 \to 0|0, 1|1 \qquad 0 \to 1|0$

Derived transducer $\mathcal{D}_{\frac{3}{2}}$



Theorem

$$\forall n \in \mathbb{N} \qquad \mathcal{D}_{\frac{3}{2}}(\mathbf{w}_n^-) = \mathbf{w}_{n+1}^-$$

Part VII

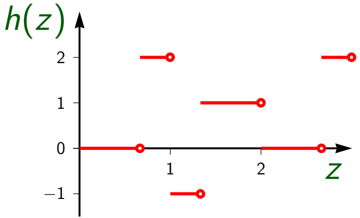
Complements

Complement 1:

The companion representation and the Mahler problem

The companion $\frac{3}{2}$-representation

$h\colon \mathbb{R}_+ \to \mathbb{Z}$ function defined by: $h(z) = 2\lfloor(\frac{3}{2})z\rfloor - 3\lfloor z\rfloor$



Proposition

h is periodic of period 2 and

$$h(z) \in C = \{-1, 0, 1, 2\}\ ,\ \forall z \in \mathbb{R}_+$$

The companion $\frac{3}{2}$-representation

$$h_n(z) = h\left((\frac{3}{2})^{n-1}z\right) = c_n$$

$$\varphi(z)\colon \mathbb{R}_+ \to C^{\mathbb{N}} \qquad\qquad \varphi(z) = \mathbf{c} = \mathbf{.}c_1c_2\cdots c_n\cdots \quad .$$

Proposition

$\forall z \in \mathbb{R}_+$, $\varphi(z)$ *is a* $\frac{3}{2}$*-representation of* $\{z\} = z - \lfloor z \rfloor$.

$\forall k \in \mathbb{N}$, $\mathbf{.}c_kc_{k+1}c_{k+2}\cdots$ *is a* $\frac{3}{2}$*-representation of* $\left\{(\frac{3}{2})^{k-1}z\right\}$.

The companion $\frac{3}{2}$-representation

$$h_n(z) = h\left((\frac{3}{2})^{n-1} z\right) = c_n$$

$$\varphi(z)\colon \mathbb{R}_+ \to C^{\mathbb{N}} \qquad \varphi(z) = \mathbf{c} = \mathbf{.}c_1 c_2 \cdots c_n \cdots \quad .$$

Proposition

$\forall z \in \mathbb{R}_+$, $\varphi(z)$ *is a* $\frac{3}{2}$*-representation of* $\{z\} = z - \lfloor z \rfloor$.

$\forall k \in \mathbb{N}$, $\mathbf{.}c_k c_{k+1} c_{k+2} \cdots$ *is a* $\frac{3}{2}$*-representation of* $\left\{(\frac{3}{2})^{k-1} z\right\}$.

Proposition

$$\forall k \in \mathbb{N}\ ,\ \ h_k(z) = 0 \quad \Longrightarrow \quad \left\{(\frac{3}{2})^{k-1} z\right\} \in [0, \tfrac{1}{3}[\ .$$

$$\forall k \in \mathbb{N}\ ,\ \ h_k(z) = 1 \quad \Longrightarrow \quad \left\{(\frac{3}{2})^{k-1} z\right\} \in [\tfrac{2}{3}, 1[\ .$$

The right converter from C^* to A^*

$C = \{-1, 0, 1, 2\}$ contains A_3 .

$$\chi_C \colon C^* \to A^* \qquad \forall w \in C^* \qquad \pi(\chi_C(w)) = \pi(w) \ .$$

Proposition

χ_C *is realised by a letter-to letter sequential right transducer.*

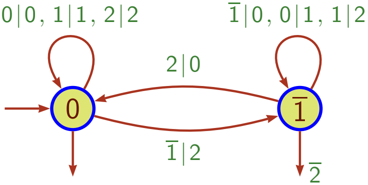
The right converter from C^* to A^*

$C = \{-1, 0, 1, 2\}$ contains A_3 .

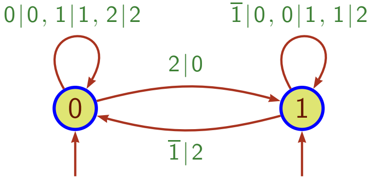
$\chi_C \colon C^* \to A^*$ $\qquad \forall w \in C^* \qquad \pi(\chi_C(w)) = \pi(w)$.

Proposition

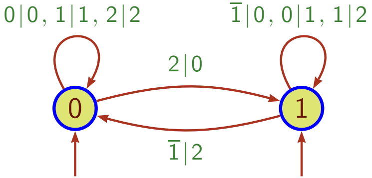
χ_C *is realised by a letter-to letter sequential right transducer.*



The left converter from C^* to A^*



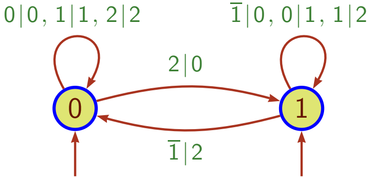
The **left** converter from C^* to A^*



Proposition

If $p \geqslant 2q - 1$ *, then the left converter has only two states.*

The left converter from C^* to A^*



Proposition

If $p \geqslant 2q-1$ *, then the left converter has only two states.*

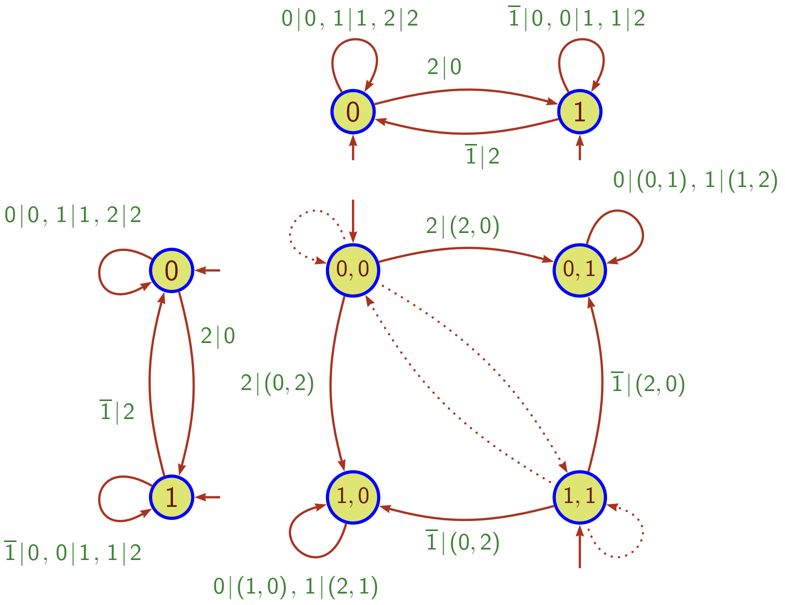
Proposition

Let $z \in [0, \omega_{\frac{3}{2}}]$ *and* $\mathbf{c}$ *its companion representation.*

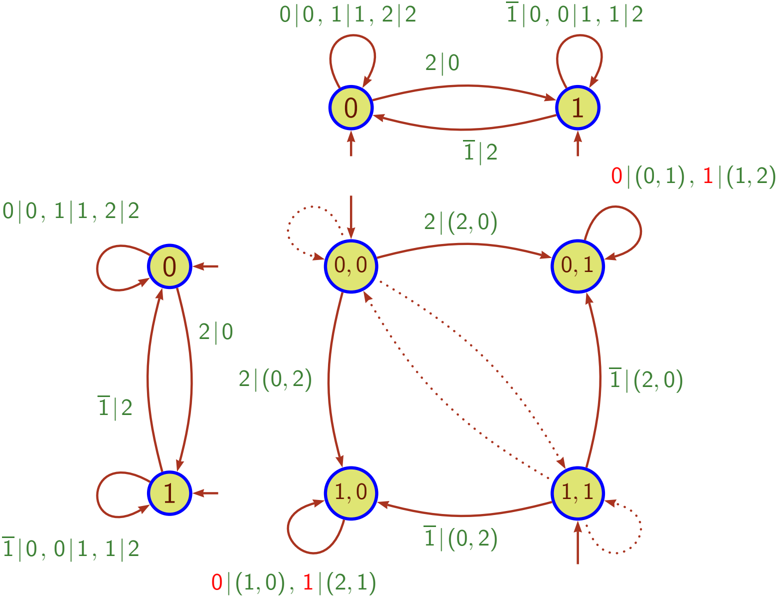
Then $\mathbf{a}$ *is a* $\frac{3}{2}$*-expansion of* z *iff*

$(\mathbf{c}, \mathbf{a})$ *is an infinite path in the left converter.*

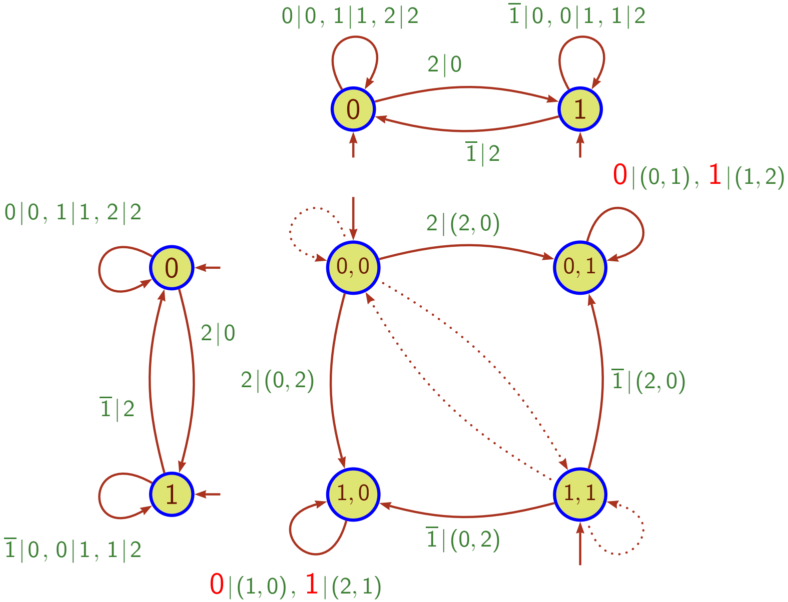
Squaring the left converter



Squaring the left converter



Squaring the left converter



Complement 2:

Languages with arbitrary rhythm

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ A *ordered* alphabet

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ $\quad$ A *ordered* alphabet

A purely periodic labelled signature

$$(\mathbf{s}, \boldsymbol{\lambda}) = (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ $\qquad$ A *ordered* alphabet

A purely periodic labelled signature

$$(\mathbf{s}, \boldsymbol{\lambda}) \;=\; (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

$\mathbf{r}$ rhythm of dir. par. (q, p) $\qquad$ $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1})$ $\quad \gamma_i \in A$

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ $\qquad$ A *ordered* alphabet

A purely periodic labelled signature

$$(\mathbf{s}, \boldsymbol{\lambda}) \;=\; (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

$\mathbf{r}$ rhythm of dir. par. (q, p) $\qquad$ $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1})$ $\quad \gamma_i \in A$

Definition

$$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$$

$\boldsymbol{\gamma} = u_0\, u_1 \cdots u_{q-1}$ factorisation induced by $\mathbf{r}$ $\qquad$ $|u_i| = r_i$

$\boldsymbol{\gamma}$ consistent with $\mathbf{r}$ $\qquad$ every u_i increasing word

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ $\quad$ A *ordered* alphabet

A purely periodic labelled signature

$$(\mathbf{s}, \boldsymbol{\lambda}) \ = \ (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

$\mathbf{r}$ rhythm of dir. par. (q, p) $\quad$ $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1})$ $\quad$ $\gamma_i \in A$

Definition

$$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$$

$\boldsymbol{\gamma} = u_0\, u_1 \cdots u_{q-1}$ factorisation induced by $\mathbf{r}$ $\quad$ $|u_i| = r_i$

$\boldsymbol{\gamma}$ consistent with $\mathbf{r}$ $\quad$ every u_i increasing word

Examples

$\mathbf{r} = (3, 1, 1)$ $\quad$ $\boldsymbol{\gamma} = 0\,1\,2\,1\,0$ $\quad$ $\boldsymbol{\gamma} = 0\,3\,5\,6\,4$ $\quad$ consistent

$\mathbf{r} = (2, 2, 1)$ $\quad$ $\boldsymbol{\gamma} = 0\,1\,2\,1\,0$ not consistent $\quad$ $\boldsymbol{\gamma} = 0\,3\,5\,6\,4$ consistent

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} = \big(0, (q \,\%\, p), (2\,q \,\%\, p), \ldots, ((p-1)\,q \,\%\, p)\big) \ .$$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} = \big(0,\, (q \,\%\, p),\, (2\,q \,\%\, p), \ldots,\, ((p-1)\,q \,\%\, p)\big) \quad .$$

Examples

$\mathbf{r}_{\frac{3}{2}} = (2, 1)$ $\gamma_{\frac{3}{2}} = 0\,2\,1$ $\mathbf{r}_{\frac{5}{3}} = (2, 2, 1)$ $\gamma_{\frac{5}{3}} = 0\,3\,1\,4\,2$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} = \big(0, (q \,\%\, p), (2q \,\%\, p), \ldots, ((p-1)\, q \,\%\, p) \big) .$$

Examples

$\mathbf{r}_{\frac{3}{2}} = (2, 1)$ $\gamma_{\frac{3}{2}} = 0\,2\,1$ $\mathbf{r}_{\frac{5}{3}} = (2, 2, 1)$ $\gamma_{\frac{5}{3}} = 0\,3\,1\,4\,2$

Proposition

$\gamma_{\frac{p}{q}}$ *is consistent with* $\mathbf{r}_{\frac{p}{q}}$

Signature of $T_{\frac{p}{q}}$

p, q coprime integers, $p > q \geqslant 1$

Theorem

The labelled signature of $T_{\frac{p}{q}}$ *is purely periodic of period* $(\mathbf{r}_{\frac{p}{q}}, \boldsymbol{\gamma}_{\frac{p}{q}})$.

Special labelling

p, q coprime integers $p > q \geqslant 1$

Special labelling

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$ rhythm of directing parameter (q, p)

Special labelling

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$ rhythm of directing parameter (q, p)

Definition

$\boldsymbol{\gamma}_{\mathbf{r}} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1}) = u_0\, u_1 \cdots u_{q-1}$

special labelling associated with $\mathbf{r}$

$$\gamma_i \in u_k\,,\ \gamma_{i+1} \in u_{k+j} \qquad \Longrightarrow \qquad \gamma_{i+1} = \gamma_i + q - j\,p$$

Special labelling

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$ rhythm of directing parameter (q, p)

Definition

$\boldsymbol{\gamma}_{\mathbf{r}} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1}) = u_0\, u_1 \cdots u_{q-1}$

special labelling associated with $\mathbf{r}$

$$\gamma_i \in u_k\,,\ \gamma_{i+1} \in u_{k+j} \qquad \Longrightarrow \qquad \gamma_{i+1} = \gamma_i + q - j\,p$$

Examples

$\mathbf{r} = (3, 1, 1)$ $\boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,6\,4\,2$ $\mathbf{r} = (4, 0, 1)$ $\boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,6\,9\,2$

$\mathbf{r} = (2, 2, 1)$ $\boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,1\,4\,2$

Special labelling

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$ rhythm of directing parameter (q, p)

Definition

$\boldsymbol{\gamma}_{\mathbf{r}} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1}) = u_0\, u_1 \cdots u_{q-1}$

special labelling associated with $\mathbf{r}$

$$\gamma_i \in u_k\,,\ \gamma_{i+1} \in u_{k+j} \qquad \Longrightarrow \qquad \gamma_{i+1} = \gamma_i + q - j\,p$$

Examples

$\mathbf{r} = (3, 1, 1) \quad \boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,6\,4\,2$ $\qquad$ $\mathbf{r} = (4, 0, 1) \quad \boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,6\,9\,2$

$\mathbf{r} = (2, 2, 1) \quad \boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,1\,4\,2$

Observation

The special labelling associated with $\mathbf{r}$ is consistent with $\mathbf{r}$

Special labelling

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r} = (r_0, r_1, \dots, r_{q-1})$ rhythm of directing parameter (q, p)

Definition

$\boldsymbol{\gamma}_{\mathbf{r}} = (\gamma_0, \gamma_1, \dots, \gamma_{p-1}) = u_0\, u_1 \cdots u_{q-1}$

special labelling associated with $\mathbf{r}$

$$\gamma_i \in u_k\,,\ \gamma_{i+1} \in u_{k+j} \qquad \Longrightarrow \qquad \gamma_{i+1} = \gamma_i + q - j\,p$$

Examples

$\mathbf{r} = (3, 1, 1)$ $\boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,6\,4\,2$ $\mathbf{r} = (4, 0, 1)$ $\boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,6\,9\,2$

$\mathbf{r} = (2, 2, 1)$ $\boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,1\,4\,2$

Proposition

$$\boldsymbol{\gamma}_{\mathbf{r}_{\frac{p}{q}}} = \boldsymbol{\gamma}_{\frac{p}{q}}$$

The tree $T_{\mathbf{r}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ rhythm of directing parameter (q, p) $\boldsymbol{\gamma}_{\mathbf{r}}$ special labelling

Definition

$T_{\mathbf{r}}$ labelled tree with labelled signature $(\mathbf{r}^{\omega}, \boldsymbol{\gamma}_{\mathbf{r}}^{\omega})$

The tree $T_{\mathbf{r}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ rhythm of directing parameter (q, p) $\qquad$ $\boldsymbol{\gamma}_{\mathbf{r}}$ special labelling

Definition

$T_{\mathbf{r}}$ labelled tree with labelled signature $(\mathbf{r}^{\omega}, \boldsymbol{\gamma}_{\mathbf{r}}^{\omega})$

Theorem

$T_{\mathbf{r}}$ *is the representation of integers in base* $\frac{p}{q}$
with non-canonical set of digits.

The tree $T_{\mathbf{r}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ rhythm of directing parameter (q, p) $\boldsymbol{\gamma}_{\mathbf{r}}$ special labelling

Definition

$T_{\mathbf{r}}$ labelled tree with labelled signature $(\mathbf{r}^{\omega}, \boldsymbol{\gamma}_{\mathbf{r}}^{\omega})$

Theorem

$T_{\mathbf{r}}$ is the representation of integers in base $\frac{p}{q}$ with non-canonical set of digits.

Corollary

$T_{\frac{p}{q}}$ is the image of $T_{\mathbf{r}}$ by a finite letter-to-letter sequential right transducer.

The tree $T_{\mathbf{r}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ rhythm of directing parameter (q, p) $\gamma_{\mathbf{r}}$ special labelling

Definition

$T_{\mathbf{r}}$ labelled tree with labelled signature $(\mathbf{r}^{\omega}, \gamma_{\mathbf{r}}^{\omega})$

$L_{\mathbf{r}}$ branch language of $T_{\mathbf{r}}$

Theorem

$L_{\mathbf{r}}$ *is the representation of integers in base* $\frac{p}{q}$ *with non-canonical set of digits.*

Corollary

$L_{\frac{p}{q}}$ *is the image of* $L_{\mathbf{r}}$ *by a finite letter-to-letter sequential right transducer.*

Complement 3:

Signature of rational languages

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

Another example: the s-morphic signatures

$$\sigma : A^* \to A^* \text{ morphism}$$

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$\sigma^1(a) = a \; b$$

Another example: the s-morphic signatures

$$\sigma : A^* \to A^* \text{ morphism}$$

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$\sigma^2(a) \; = \; a \;\; b \;\; a$$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$\sigma^3(a) \; = \; a \;\; b \;\; a \;\; a \;\; b$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$\sigma^4(a) \;=\; a \;\; b \;\; a \;\; a \;\; b \;\; a \;\; b \;\; a$$

Another example: the s-morphic signatures

$$\sigma : A^* \to A^* \text{ morphism}$$

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$\sigma^5(a) \ = \ a \ b \ a \ a \ b \ a \ b \ a \ a \ b \ a \ a \ b$$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$\sigma^6(a) \; = \; a \;\; b \;\; a \;\; a \;\; b \;\; a \;\; b \;\; a \;\; a \;\; b \;\; a \;\; a \;\; b \;\; a \;\; b \;\; a \;\; a \;\; b \;\; a \;\; b \;\; a$$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = ab \qquad \sigma(b) = a$$

$$\sigma^\omega(a) \ = \ a \ b \ a \ a \ b \ a \ b \ a \ a \ b \ a \ a \ b \ a \ b \ a \ a \ b \ a \ b \ a \cdots$$

Another example: the s-morphic signatures

$$\sigma\colon A^* \to A^* \text{ morphism}$$

$f_\sigma\colon A^* \to D^*$ morphism

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$f_\sigma(a) = |\sigma(a)| = 2 \qquad f_\sigma(b) = |\sigma(b)| = 1$$

$$\begin{array}{rcccccccccccccccccccccc} \sigma^\omega(a) & = & a & b & a & a & b & a & b & a & a & b & a & a & b & a & b & a & a & b & a & b & a \cdots \\ f_\sigma(\sigma^\omega(a)) & = & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 \cdots \end{array}$$

Another example: the s-morphic signatures

$$\sigma : A^* \to A^* \text{ morphism}$$

$$f_\sigma : A^* \to D^* \text{ morphism} \qquad g : A^* \to B^* \text{ morphism}$$

$$\sigma(a) = ab \qquad \sigma(b) = a$$

$$f_\sigma(a) = |\sigma(a)| = 2 \qquad f_\sigma(b) = |\sigma(b)| = 1$$

$$g(a) = 01 \qquad g(b) = 0$$

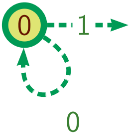
$$\begin{aligned}
\sigma^\omega(a) &= a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ b\ a \cdots \\
f_\sigma(\sigma^\omega(a)) &= 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2 \cdots \\
g(\sigma^\omega(a)) &= 01\ 0\ 01\ 01\ 0\ 01\ 0\ 01\ 01\ 0\ 01\ 01\ 0\ 01\ 0\ 01\ 01\ 0\ 01\ 0\ 01 \cdots
\end{aligned}$$

Another example: the s-morphic signatures

$$\mathbf{s} = \; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \cdots$$

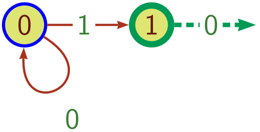
$$\boldsymbol{\lambda} = 01 \;\; 0 \;\; 0101 \;\; 0 \;\; 01 \;\; 0 \;\; 0101 \;\; 0 \;\; 0101 \;\; 0 \;\; 01 \;\; 0 \cdots$$

Another example: the s-morphic signatures



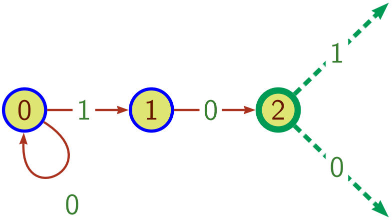
$$\mathbf{s} = \color{red}{2}\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 2\ 1 \cdots$$
$$\boldsymbol{\lambda} = \color{red}{01}\ 0\ 0101\ 0\ 01\ 0\ 0101\ 0\ 0101\ 0\ 01\ 0 \cdots$$

Another example: the s-morphic signatures



$$\mathbf{s} = 2\ \ 1\ \ 2\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 2\ \ 1\ \ 2\ \ 2\ \ 1\ \ 2\ \ 1\ \cdots$$
$$\boldsymbol{\lambda} = 01\ \ 0\ \ 0101\ \ 0\ \ 01\ \ 0\ \ 0101\ \ 0\ \ 0101\ \ 0\ \ 01\ \ 0\ \cdots$$

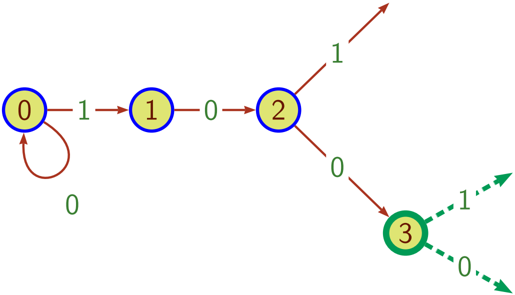
Another example: the s-morphic signatures



$$\mathbf{s} = \; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \cdots$$

$$\boldsymbol{\lambda} = 01 \;\; 0 \;\; 01\, 01 \;\; 0 \;\; 01 \;\; 0 \;\; 01\, 01 \;\; 0 \;\; 01\, 01 \;\; 0 \;\; 01 \;\; 0 \;\; \cdots$$

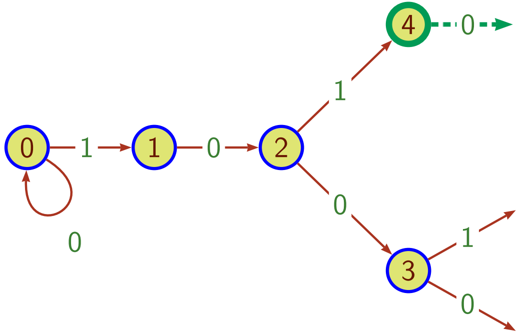
Another example: the s-morphic signatures



$$\mathbf{s} = \;2\;\;1\;\;2\;\;\color{red}{2}\;\;1\;\;2\;\;1\;\;2\;\;2\;\;1\;\;2\;\;2\;\;1\;\;2\;\;1\;\cdots$$

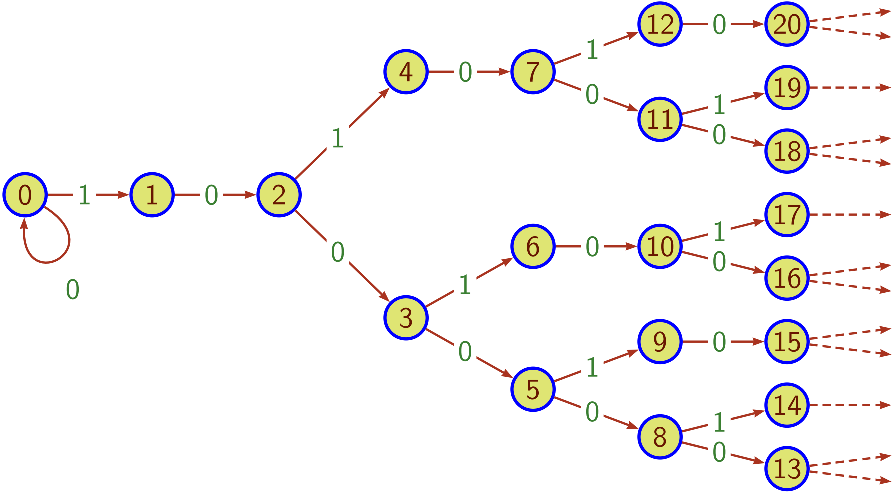
$$\boldsymbol{\lambda} = 01\;\;0\;\;01\color{red}{01}\;\;0\;\;01\;\;0\;\;0101\;\;0\;\;0101\;\;0\;\;01\;\;0\;\cdots$$

Another example: the s-morphic signatures



$$\mathbf{s} = 2\ 1\ 2\ 2\ \color{red}{1}\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 2\ 1 \cdots$$
$$\boldsymbol{\lambda} = 01\ 0\ 0101\ \color{red}{0}\ 01\ 0\ 0101\ 0\ 0101\ 0\ 01\ 0 \cdots$$

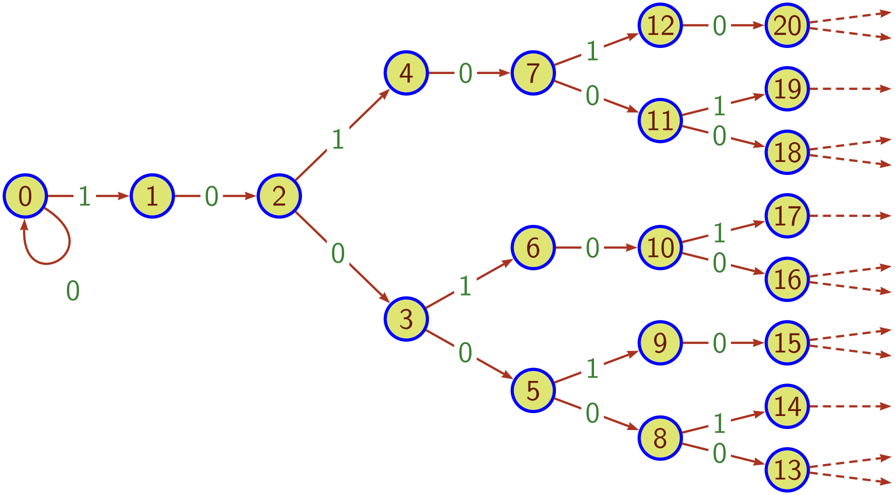
Another example: the s-morphic signatures



$$\mathbf{s} = 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 2\ 1\ \cdots$$

$$\boldsymbol{\lambda} = 01\ 0\ 0101\ 0\ 01\ 0\ 0101\ 0\ 0101\ 0\ 01\ 0\ \cdots$$

Another example: the s-morphic signatures



$$T = \{0,1\}^* \setminus \{0,1\}^* 11 \{0,1\}^*$$

Another example: the s-morphic signatures

Theorem (Cobham 72, Rigo–Maes 02, M.–S. 14)

A prefix-closed language is regular iff its labelled signature is s-morphic.