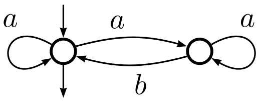
Lecture I — Exercises

Unless stated otherwise, the alphabet A is $A = \{a, b\}$.

1. **Semiring structure.** Is $\mathbb{M} = \langle \mathbb{N}, \mathsf{max}, +, 0, 0 \rangle$ a semiring?

2. **Positive semiring.** Give an example of a semiring in which the sum of any two non-zero elements is non-zero but which *is not* positive. [*Hint: consider a sub-semiring of* $\mathbb{N}^{2\times 2}$.]

3. **Example of $\mathbb{N}$-automaton.** (a) Compute the coefficient of $a^3 b a^2 b a$ in the series realised by the $\mathbb{N}$-automaton:



(b) Give the general formula for the coefficient of every word of A^*.

4. **Examples of $\mathbb{N}\mathsf{min}$, $\mathbb{N}\mathsf{max}$-automata.** Let $\mathcal{E}_1$ be the $\mathbb{N}\mathsf{min}$-automaton over $\{a\}^*$ shown in Fig. 1 (a) and $\mathcal{E}_2$ the $\mathbb{N}\mathsf{max}$-automaton shown in the same figure. Similarly, let $\mathcal{E}_3$ and $\mathcal{E}_4$ be the $\mathbb{N}\mathsf{min}$ and $\mathbb{N}\mathsf{max}$-automata shown in Fig. 1 (b).

Give a formula for $\langle\!|\mathcal{E}_1|\!, a^n\rangle$, $\langle\!|\mathcal{E}_2|\!, a^n\rangle$, $\langle\!|\mathcal{E}_3|\!, a^n\rangle$, and $\langle\!|\mathcal{E}_4|\!, a^n\rangle$.

(a) The automata $\mathcal{E}_1$ and $\mathcal{E}_2$ (b) The automata $\mathcal{E}_3$ and $\mathcal{E}_4$

Figure 1: 'Four' 'tropical' automata

5. **A $\mathbb{Z}$-automaton.** Build a $\mathbb{Z}$-automaton $\mathcal{D}_1$ such that $\langle\!|\mathcal{D}_1|\!, w\rangle = |w|_a - |w|_b$, for every w in A^*.

6. **Support of $\mathbb{Z}$-automata.** Give an example of a $\mathbb{Z}$-automaton $\mathcal{A}$ such that the inclusion $\mathsf{supp}\ (|\mathcal{A}|) \subseteq |\mathsf{supp}\ \mathcal{A}|$ is strict.

7. **Automata construction.** Let $\underline{a^*}$ be the characteristic $\mathbb{N}$-series of a^* : $\underline{a^*} = \sum_{n\in\mathbb{N}} a^n$. Give an 'automatic' proof (that is, by means of automata constructions) for:

$$(\underline{a^*})^2 = \sum_{n\in\mathbb{N}} (n+1)\, a^n \quad .$$

8. **Shortest run and $\mathbb{N}\mathsf{min}$-automata.** Build a $\mathbb{N}\mathsf{min}$-automaton $\mathcal{F}_1$ such that, for every w in A^*, $\langle\!|\mathcal{F}_1|\!, w\rangle$ is the minimal length of runs of 'a' 's in w, that is, if $w = a^{n_0}\, b\, a^{n_1}\, b \cdots a^{n_{k-1}}\, b\, a^{n_k}$, then $\langle\!|\mathcal{F}_1|\!, w\rangle = \mathsf{min}\{n_0, n_1, \ldots, n_k\}$.

9. **Identification of a $\mathbb{Q}$-automaton.** Show that the final function of the $\mathbb{Q}$-automaton $\mathcal{Q}_2$ over $\{a\}^*$ depicted on the right in Figure 2 (where every transition is labelled by $a\,|\,1$) can be specified in such a way the result is equivalent to $\mathcal{Q}_1$ depicted on the left.

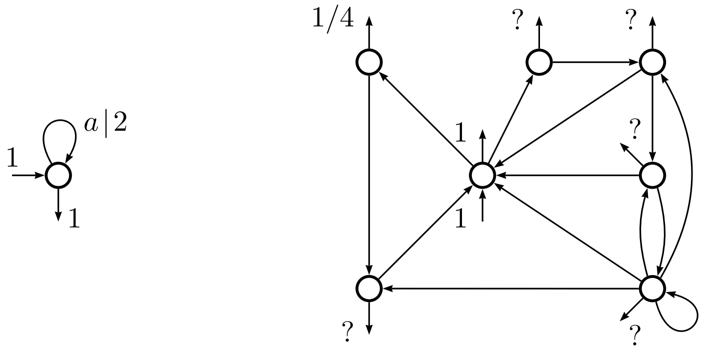


Figure 2: Two $\mathbb{Q}$-automata

10. **Ambiguous automata.** Show that it is decidable whether a Boolean automaton is unambiguous or not. [*Hint: Note that this is* not *a result nor a proof on weighted automata but on Boolean automata. It is put here in view of Example 49.*]

11. **Representation with finite image.** Let s be a $\mathbb{K}$-recognisable series of A^*, realised by a representation $\langle I, \mu, T \rangle$ of dimension Q. Show that if $\mu(A^*)$ is a finite submonoid of $\mathbb{K}^{Q \times Q}$, then, for every k in $\mathbb{K}$ the set $s^{-1}(k) = \{w \in A^* \mid \langle s, w \rangle = k\}$ is a recognisable language of A^*.

12. **Support of $\mathbb{Z}$-rational series.** (a) Give an example of a $\mathbb{Z}$-rational series over A^* whose support is not a recognisable language of A^*.

(b) Give an example of a $\mathbb{Z}$-rational series over A^* which is an $\mathbb{N}$-series (that is, all coefficients are non-negative) and which is not an $\mathbb{N}$-rational series over A^*.

13. **Support of $\mathbb{Z}$-rational series.** (a) Prove that the support of an $\mathbb{N}$-rational series over A^* is a recognisable language of A^*.

(b) Let s be in $\mathbb{N}\mathrm{Rec}\, A^*$. Prove that for any k in $\mathbb{N}$, the sets

$$s^{-1}(k) = \{w \in A^* \mid \langle s, w \rangle = k\} \quad \text{and} \quad s^{-1}(k + \mathbb{N}) = \{w \in A^* \mid \langle s, w \rangle \geqslant k\}$$

are recognisable languages of A^*.

(c) Give an example of a $\mathbb{Z}$-rational series s over A^* such that there exists an integer z such that $s^{-1}(z)$ is not a recognisable language of A^*.

14. **Support of $\mathbb{Z}\mathsf{min}$-rational series.** (a) Let s be a $\mathbb{N}\mathsf{min}$-rational series over A^*. Prove that for any k in $\mathbb{N}$, the sets

$$s^{-1}(k) = \{w \in A^* \mid \langle s, w \rangle = k\} \qquad \text{and} \qquad s^{-1}(k + \mathbb{N}) = \{w \in A^* \mid \langle s, w \rangle \geqslant k\}$$

are recognisable languages of A^*.

(b) Give an example of a $\mathbb{Z}\mathsf{min}$-rational series s over A^* such that there exists an integer z such that $s^{-1}(z)$ is not a recognisable language of A^*.

15. **Recognisable series in direct product of free monoids.** Let $\mathbb{K}$ be a commutative semiring. The two semirings $\mathbb{K}\langle\!\langle A^* \rangle\!\rangle$ and $\mathbb{K}\langle\!\langle B^* \rangle\!\rangle$ are canonically subalgebras of $\mathbb{K}\langle\!\langle A^* \times B^* \rangle\!\rangle$; the injection is induced by

$$u \mapsto (u, 1_{B^*}) \qquad \text{and} \qquad v \mapsto (1_{A^*}, v) \enspace ,$$

for all u in A^* and all v in B^*. Modulo this identification, a product $(k\,u)\,(h\,v)$ is written $k\,h\,(u,v)$ and the extension by linearity of this notation gives the following definition.

Definition 1. Let s be in $\mathbb{K}\langle\!\langle A^*\rangle\!\rangle$ and t be in $\mathbb{K}\langle\!\langle B^*\rangle\!\rangle$. The *tensor product* of s and t, written $s \otimes t$, is the series of $\mathbb{K}\langle\!\langle A^* \times B^*\rangle\!\rangle$ defined by:

$$\forall (u,v) \in A^* \times B^* \qquad \langle s \otimes t, (u,v)\rangle = \langle s,u\rangle\,\langle t,v\rangle \ .$$

On the other hand, $\mathbb{K}$-*recognisable series* over a non-free monoid M are defined, exactly as the $\mathbb{K}$-recognisable series over a free monoid, as the series realised by a $\mathbb{K}$-representation $\langle\, I, \mu, T\,\rangle$, where μ is a morphism from M into $\mathbb{K}^{Q\times Q}$.

Establish:

Proposition 2. *A series s of $\mathbb{K}\langle\!\langle A^* \times B^*\rangle\!\rangle$ is recognisable if and only if there exists a finite family $\{r_i\}_{i\in I}$ of series of $\mathbb{K}\mathrm{Rec}\,A^*$ and a finite family $\{t_i\}_{i\in I}$ of series of $\mathbb{K}\mathrm{Rec}\,B^*$ such that*

$$s = \sum_{i\in I} r_i \otimes t_i \ .$$

16. **Distance on the semirings of series.**

A *distance* on any set S is a map $\mathbf{d}\colon S \times S \to \mathbb{R}_+$ with the three properties: for all x, y and z in S it holds:

(i) *symmetry*: $\mathbf{d}(x,y) = \mathbf{d}(y,x)$;

(ii) *positivity*: $\mathbf{d}(x,y) = 0 \Leftrightarrow x = y$;

(iii) *triangular inequality*: $\mathbf{d}(x,z) \leqslant \mathbf{d}(x,y) + \mathbf{d}(y,z)$.

If (iii) is replaced by the stronger property:

(iv) *triangular inequality*: $\mathbf{d}(x,z) \leqslant \mathsf{max}(\mathbf{d}(x,y), \mathbf{d}(y,z))$,

then $\mathbf{d}$ is said to be an *ultrametric distance*.

(a) Show that the function defined on S by

$$\forall x,y \in S \qquad \mathbf{d}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

is an ultrametric distance. We call it the *discrete distance* on S.

Classically, a sequence $\left(s_n\right)_{n\in\mathbb{N}}$ of elements of S *converges* to s in S for the distance $\mathbf{d}$ if:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \qquad \mathbf{d}(s_n, s) < \varepsilon \ .$$

In this way, a distance $\mathbf{d}$ defines a *topology* on S.

(b) Show that if S is equipped with the discrete distance, the only convergent sequences are the ultimately stationnary sequences.

Two distances on S are *equivalent* if the same sequences converge, that is, $\mathbf{d}$ and $\mathbf{d}'$ are equivalent if for any sequence $s = \left(s_n\right)_{n\in\mathbb{N}}$, s converges for $\mathbf{d}$ if and only if it converges for $\mathbf{d}'$.

(c) Show that one can always assume that *a distance is bounded by* 1, that is, if $\mathbf{d}$ is a distance on S, the function $\mathbf{f}$ defined by

$$\forall x,y \in S \qquad \mathbf{f}(x,y) = \mathsf{inf}\{\mathbf{d}(x,y), 1\}$$

is a distance, equivalent to $\mathbf{d}$.

(d) Let $\mathbf{d}$ and $\mathbf{d}$' be two distances on S. Show that if there exist two constant C and D in $\mathbb{R}_+ \setminus \{0\}$ such that

$$\forall x, y \in S \qquad C\,\mathbf{d}\,(x,y) \leqslant \mathbf{d}'\,(x,y) \leqslant D\,\mathbf{d}\,(x,y)$$

then $\mathbf{d}$ and $\mathbf{d}$' are equivalent. Is this condition necessary for $\mathbf{d}$ and $\mathbf{d}$' be equivalent?

Let $\mathbb{K}$ be a semiring. For s and t in $\mathbb{K}\langle\langle A^*\rangle\rangle$, let $\mathbf{e}(s,t)$ be the *gap between s and t*, defined as the minimal length of words on which s and t are different:

$$\mathbf{e}(s,t) = \mathsf{min}\,\{n \in \mathbb{N} \mid \exists w \in A^*, \quad |w| = n \text{ and } \langle s, w\rangle \neq \langle t, w\rangle\} \quad .$$

The gap is a generalisation of the notion of *valuation* of a series. The valuation $\mathbf{v}(s)$ of s in $\mathbb{K}\langle\langle A^*\rangle\rangle$ is defined by:

$$\mathbf{v}(s) = \mathbf{e}(s,0) = \mathsf{min}\,\{|w| \mid \langle s,w\rangle \neq 0\} = \mathsf{min}\,\{|w| \mid w \in \mathsf{supp}\ s\} \quad .$$

Conversely, and if $\mathbb{K}$ is a *ring*, $\mathbf{e}(s,t) = \mathbf{v}(s-t)$.

(e) Show that the map defined by

$$\forall s,t \in \mathbb{K}\langle\langle A^*\rangle\rangle \qquad \mathbf{d}'\,(s,t) = 2^{-\,\mathbf{e}(s,t)} \tag{0.1}$$

is an ultrametric distance on $\mathbb{K}\langle\langle A^*\rangle\rangle$, bounded by 1.

(f) Let $\mathbf{c}$ be a distance on $\mathbb{K}\langle\langle A^*\rangle\rangle$, bounded by 1. Show that the map defined by

$$\forall s,t \in \mathbb{K}\langle\langle A^*\rangle\rangle \qquad \mathbf{d}\,(s,t) = \frac{1}{2}\sum_{n\in\mathbb{N}}\left(\frac{1}{2^n}\,\mathsf{max}\,\{\mathbf{c}\,(\langle s,w\rangle, \langle t,w\rangle) \mid |w| = n\}\right) \tag{0.2}$$

is a distance on $\mathbb{K}\langle\langle A^*\rangle\rangle$, bounded by 1.

(g) Show that, whatever the distance $\mathbf{c}$, $\mathbf{d}\,(s,t) \leqslant \mathbf{d}'\,(s,t)$ holds.

(h) Show that if $\mathbf{c}$ is the discrete distance, then $\mathbf{d}'\,(s,t) \leqslant 2\,\mathbf{d}\,(s,t)$ holds, hence that (0.1) and (0.2) define two equivalent distances on $\mathbb{K}\langle\langle A^*\rangle\rangle$ if $\mathbb{K}$ is equipped with the discrete distance.

(i) Show that the topology defined by $\mathbf{d}$ on $\mathbb{K}\langle\langle A^*\rangle\rangle$ is the topology of the simple convergence.

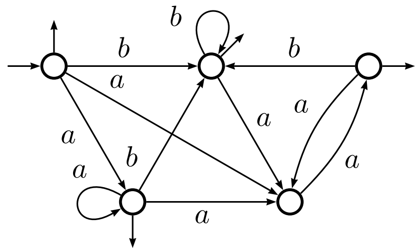
(j) Show that if $\mathbb{K}$ is a topological semiring, then so are $\mathbb{K}^{Q\times Q}$ (Q finite) and $\mathbb{K}\langle\langle A^*\rangle\rangle$.

(k) Let $\left(s_n\right)_{n\in\mathbb{N}}$ and $\left(t_n\right)_{n\in\mathbb{N}}$ be two sequences of series in the topological semiring $\mathbb{K}\langle\langle A^*\rangle\rangle$. Verify that $\left(s_n + t_n\right)_{n\in\mathbb{N}}$ or $\left(s_n\, t_n\right)_{n\in\mathbb{N}}$ may be convergent sequences, without $\left(s_n\right)_{n\in\mathbb{N}}$ or $\left(t_n\right)_{n\in\mathbb{N}}$ being convergent sequences.

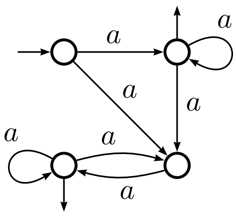
Lecture II — Exercises

Unless stated otherwise, the alphabet A is $A = \{a, b\}$.

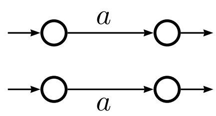
1. Compute the (minimal) quotient of the following $\mathbb{B}$-automaton:



2. Let $\mathcal{D}_1$ be the $\mathbb{B}$-automaton below. Compute the (minimal) quotient of $\mathcal{D}_1$, the co-quotient of $\mathcal{D}_1$, the co-quotient of the quotient of $\mathcal{D}_1$, *etc.*



3. Calculate all the quotients and all the co-quotients of the $\mathbb{N}$-automaton:

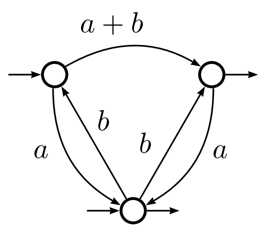


4. <**Coloured Transition Lemma.** Establish the following statement:

Let $\mathcal{A}$ be a (Boolean) automaton on a monoid M the transitions of which are coloured in red or in blue. Then, the set of labels of computations of $\mathcal{A}$ that contain at least one red transition is a rational set (of M).

5. Show that any $\mathbb{Z}$-rational series is the difference of two $\mathbb{N}$-rational series.

6. Construct the Schützenberger covering $\mathcal{S}$ of the following $\mathbb{B}$-automaton $\mathcal{A}$.



How many S-immersions are there in this covering (that is, how many sub-automata $\mathcal{T}$ of $\mathcal{S}$ that are unambiguous and equivalent to $\mathcal{A}$)?

7. Compute the Schützenberger covering of the $\mathbb{B}$-automaton $\mathcal{B}_1$ of the Figure 1.

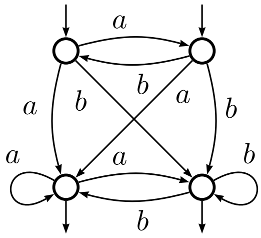


Figure 1: The automaton $\mathcal{B}_1$

8. **Quotients and product of automata.** Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three $\mathbb{K}$-automata on A^*. Show that if $\mathcal{B}$ is a quotient of $\mathcal{A}$, then $\mathcal{B}\otimes\mathcal{C}$ is a quotient of $\mathcal{A}\otimes\mathcal{C}$.

9. **Quotients and co-quotients of the $\mathcal{C}_n$.**

The $\mathbb{N}$-automaton $\mathcal{C}_1$ over $\{a, b\}^*$ shown at Figure 2 (a) associates with every word w the integer $\overline{w}$ the binary representation of which is w when a is replaced by the digit 0 and b by 1.

Let $\mathcal{C}_2$ be the tensorial square of $\mathcal{C}_1$: $\mathcal{C}_2 = \mathcal{C}_1\otimes\mathcal{C}_1$; $\mathcal{V}_2$, shown at Figure 2 (b), is the minimal quotient of $\mathcal{C}_2$ and $\mathcal{V}'_2$, shown at Figure 2 (c), is the minimal co-quotient of $\mathcal{C}_2$.

(a) Compute the minimal quotient $\mathcal{V}_3$ and the minimal co-quotient $\mathcal{V}'_3$ of $\mathcal{C}_3 = \mathcal{C}_2\otimes\mathcal{C}_1$.

(b) Compute the minimal co-quotient $\mathcal{V}'_4$ of $\mathcal{C}_4 = \mathcal{C}_3\otimes\mathcal{C}_1$. Compare with $\mathcal{V}'_3$.

(c) Generalising the above computation, compute the minimal co-quotient $\mathcal{V}'_{n+1}$ of $\mathcal{C}_{n+1} = \mathcal{C}_n\otimes\mathcal{C}_1$, for every n.

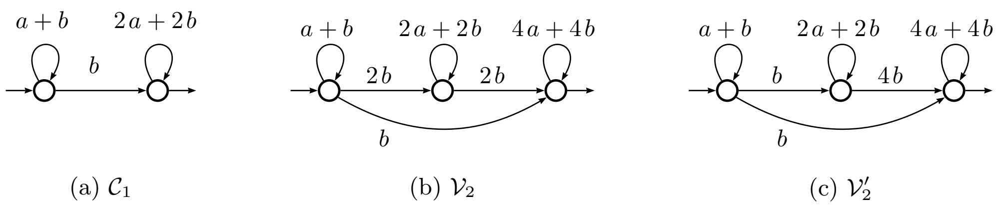


Figure 2: Three $\mathbb{N}$-automata

10. **Conjugacy of an automaton and its determinisation.**

(a) Let $\mathcal{A}_1$ be the (Boolean) automaton of Figure 3 and $\widehat{\mathcal{A}_1}$ its determinisation. Verify that $\widehat{\mathcal{A}_1} \xRightarrow{X_1} \mathcal{A}_1$ holds, with

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} .$$

(b) Generalisation. Let $\mathcal{A}$ be a (Boolean) automaton and $\widehat{\mathcal{A}}$ its determinisation. Show that there exists an Boolean matrix X such that $\widehat{\mathcal{A}} \xRightarrow{X} \mathcal{A}$.

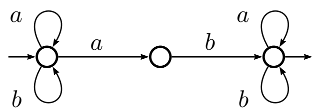


Figure 3: L'automate $\mathcal{A}_1$

11. **Automata with bounded ambiguity and the Schützenberger covering.** In the sequel, $\mathcal{A}$ is a Boolean automaton, $\widehat{\mathcal{A}}$ its determinisation, and $\mathcal{S}$ its Schützenberger covering.

Definition 3. We call *concurrent transition set* of $\mathcal{S}$ a set of transitions which

(i) have the same destination (final extremity),

(ii) are mapped onto the same transition of $\widehat{\mathcal{A}}$.

Two transitions of $\mathcal{S}$ are called *concurrent* if they belong to the same concurrent transition set.

We also set the folllowing definition:

Definition 4. An automaton $\mathcal{A}$ over A^* is of *bounded ambiguity* if there exists an integer k such that every word w in $|\mathcal{A}|$ is the label of at most k distinct computations. The smallest such k is the *ambiguity degree* of $\mathcal{A}$.

(a) What can be said of an automaton whose Schützenberger covering contains no concurrent transitions?

(b) Show that there exists a computation in $\mathcal{S}$ which contains two transitions of the same concurrent transition set if and only if there exists a concurrrent transition which belongs to a circuit.

(c) Let $p \xrightarrow{a} s$ and $q \xrightarrow{a} s$ be two concurrent transitions of $\mathcal{S}$ and

$$c \;:=\; \xrightarrow[\mathcal{S}]{} i \xrightarrow[\mathcal{S}]{x} p \xrightarrow[\mathcal{S}]{a} s \xrightarrow[\mathcal{S}]{y} q \xrightarrow[\mathcal{S}]{a} s \xrightarrow[\mathcal{S}]{z} t \xrightarrow[\mathcal{S}]{}$$

a computation of $\mathcal{S}$ where i is an initial state and t a final state. Show that $w = x\,a\,y\,a\,z$ is the label of at least two computations of $\mathcal{A}$.

(d) Prove that an automaton $\mathcal{A}$ is of bounded ambiguity if and only if no concurrent transition of its Schützenberger covering belongs to a circuit.

(e) Check that $\mathcal{B}_1$ of Figure 1 is of bounded ambiguity.

(f) Give a bound on the ambiguity degree of an automaton as a function of the cardinals of the concurrent transition sets of its Schützenberger covering.

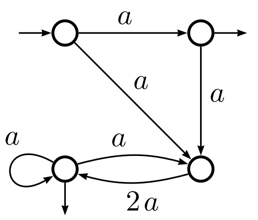
Compute that bound in the case of $\mathcal{B}_1$.

(g) Infer from the above the complexity of an algorithm which decide if an automaton is of bounded ambiguity.

Lecture III — Exercises

Unless stated otherwise, the alphabet A is $A = \{a, b\}$.

1. Compute the reduced representation of the following $\mathbb{N}$-automaton.



2. Let $\mathcal{A}_1$ be the $\mathbb{Q}$-automaton on $\{a\}^*$ shown at Figure 1 (the unique letter a of the alphabet is not shown on the transitions of the figure). Compute a reduced automaton, equivalent to $\mathcal{A}_1$.

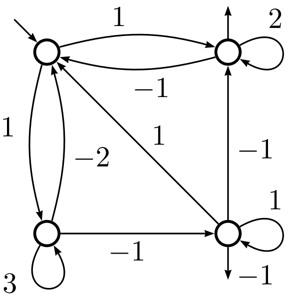
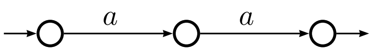


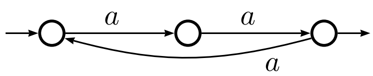
Figure 1: The $\mathbb{Q}$-automaton $\mathcal{A}_1$

3. Consider the minimal (Boolean) automaton of $\{a^n \mid n \equiv 0, 1, 2, 4 \pmod 7\}$ as an automaton with multiplicity in $\mathbb{Z}/2\mathbb{Z}$ and reduce it. Comment.

4. Let $\mathbb{F}$ be a field. Show that two $\mathbb{F}$-recognisable series over A^* are equal if and only if they coincide on all the words of length less than the sum of the dimensions of the representations which realise them.

Show the bound is sharp. [*Hint: consider the following two automata.*]





5. **Discriminating length.** We call the *discriminating length* between two non-equivalent (Boolean) automata $\mathcal{A}$ and $\mathcal{B}$ the length of a shortest word which is accepted by one and not the other. We write $L_{\mathsf{d}}(n, m)$ (resp. $L_{\mathsf{nd}}(n, m)$) for the maximum of the discriminating lengths when $\mathcal{A}$ and $\mathcal{B}$ have respectively n and m states and are deterministic (resp. and are non-deterministic).

(a) With methods relevant to Boolean automata, show that $L_{\mathsf{d}}(n, m) \leqslant n\,m$.

(b) Compute $L_{\mathsf{d}}(n, m)$.

(c) Give an upper bound for $L_{\mathsf{nd}}(n, m)$.

Lecture IV — Exercises

Unless stated otherwise, the alphabet A is $A = \{a, b\}$.

1. **Orders.** The alphabet A is *totally ordered* and this order is denoted by $\leqslant$.

The *lexicographic order*, denoted by $\preccurlyeq$, extends the order on A to an order on A^* and is defined as follows. Let v and w be two words in A^* and u their *longest common prefix.* Then, $v \preccurlyeq w$ if $v = u$ or, if $v = u\,a\,s$, $w = u\,b\,t$ with a and b in A, then $a < b$.

(a) Give a finite transducer over $A^* \times A^*$ which realises $\preccurlyeq$, that is, which asssociates with every word u of A^* the set of words which are equal to or greater than u.

The *radix order* (also called the *genealogical order* or the *short-lex order*), denoted by $\sqsubseteq$, is defined as follows: $v \sqsubseteq w$ if $|v| < |w|$ or $|v| = |w|$ and $v \preccurlyeq w$.

(b) Give a finite transducer over $A^* \times A^*$ which realises $\sqsubseteq$,

For every language L of A^*, we denote by $\mathsf{minlg}\,(L)$ (resp. $\mathsf{Maxlg}\,(L)$) the set of words of L which have no smaller (resp. no greater) words in L of *the same length* in the lexicographic order.

(c) Show that if L is a rational language, so are $\mathsf{minlg}\,(L)$ and $\mathsf{Maxlg}\,(L)$.

2. **Number representation.**

Let $A_2 = \{0, 1\}$ and $A_3 = \{0, 1, 2\}$ be two alphabets of digits.

The alphabet A_3 can be first considered as a non-canonical alphabet for the representation of integers in base 2: $\overline{1\,2} = 4$, $\overline{2\,0\,1} = 9$, *etc.*

Let $\nu_2 \colon A_3^* \to A_2^*$ be the *normalisation* in base 2, that is, the relation which associates with a word of A_3^* the word of A_2^* which represents the same integer in base 2.

(a) Give a transducer which realises ν_2. Comment.

Let $\varphi \colon A_2^* \to A_3^*$ be the function which maps the binary representation of every integer onto its representation in base 3, *e.g.* $\varphi(1\,0\,0\,0) = 2\,2$.

(b) Show that φ is not a rational relation.

3. **Operation on numbers.**

(a) Give a transducer which realises the multiplication by 9 on the integers written in binary representation, that is, the relation $\tau \colon A_2^* \to A_2^*$ such that $\overline{\tau(w)} = 9 \cdot \overline{w}$.

(b) Let $\mu \colon A_2^* \times A_2^* \to A_2^*$ be the relation which realises the multiplication, that is, such that $\mu(u, v) = w$ where $\overline{w} = \overline{u} \cdot \overline{v}$.

Show that μ is not a rational relation.

4. **Map equivalence of a morphism.**

Let $\varphi_1\colon \{a,b,c\}^* \to \{x,y\}^*$ be the morphism defined by:

$$\varphi_1(a) = x\ , \quad \varphi_1(b) = y\,x\ , \quad \varphi_1(c) = x\,y\ .$$

(a) Give a subnormalised transducer which realises φ_1.

(b) Give a subnormalised transducer which realises ${\varphi_1}^{-1}$.

(c) Compute a subnormalised transducer which realises ${\varphi_1}^{-1} \circ \varphi_1$.

5. **Iteration Lemma.** Let $\theta\colon A^* \to B^*$ be a rational relation.

(a) Show that there exists an integer N such that for every pair (u,v) in $\widehat{\theta}$ whose length[1] is greater than N, there exists a factorisation:

$$(u,v) = (s,t)\,(x,y)\,(w,z)$$

such that: (i) $1 \leqslant |x| + |y| \leqslant N$ and (ii) $(u,v) = (s,t)\,(x,y)^*(w,z) \subseteq \widehat{\theta}\ .$

(b) Show that the *mirror* function $\rho\colon A^* \to A^*$:

$$\rho(a_1\,a_2 \cdots\, a_n) = a_n\,a_{n-1} \cdots\, a_1\ \ ,$$

is not a rational relation.

[*Hint: Let* $K = a^*b^*$, $L = b^*a^*$. *Consider the relation* $\pi = \iota_L \circ \rho \circ \iota_K$ *and apply the Iteration Lemma to a pair* $(a^N b^N, b^N a^N)$.]

6. **Conjugacy.** Let $\mathsf{Conj}\colon A^* \to A^*$ be the relation which associates with every word w the set of its *conjugates*: $\mathsf{Conj}(w) = \{v\,u \mid u,v \in A^* \quad u\,v = w\}$.

(a) Show that if L is a rational language, then so is $\mathsf{Conj}(L)$.

(b) Give a transducer which associates with every word w of $\{a,b\}^*$ the word obtained by moving the first letter of w to its end.

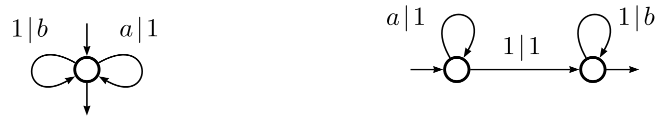
(c) Compose this transducer with itself.

(d) Show that Conj is not a rational relation.

[1] The length of a pair is the sum of the lengths of its components.

Lecture V — Exercises

1. Apply the construction of the proof of Theorem 3 in order to build real-time transducers from the two transducers below which realise the universal relation on $\{a\}^* \times \{b\}^*$.



(a) $\mathcal{U}_1$ (b) $\mathcal{U}_2$

2. Give a realisation by representation of the following relations:

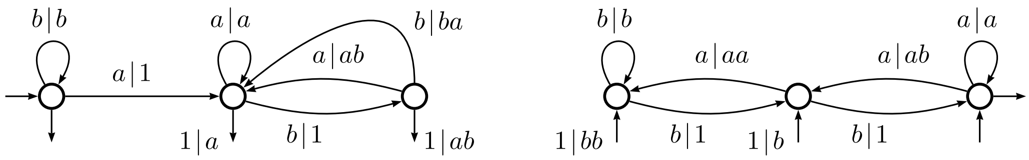
(a) the complement of the identity; (b) the lexicographic order; (c) the radix order.

3. **Finite and infinite components of a rational relation.** Let $\tau\colon A^* \to B^*$ be a relation. The *finite* and *infinite components* τ_{f} and τ_∞ of τ are defined by:

$$\tau_{\mathsf{f}}(w) = \begin{cases} \tau(w) & \text{if } \|\tau(w)\| \text{ is finite} \\ \emptyset & \text{otherwise} \end{cases} \quad \text{et} \quad \tau_\infty(w) = \begin{cases} \emptyset & \text{if } \|\tau(w)\| \text{ is finite} \\ \tau(w) & \text{otherwise} \end{cases}$$

Show that if τ is rational, then τ_{f} and τ_∞ are rational and effectively computable from τ.

4. **Fibonacci reduction.** Give a transducer which realises the composition of the relations realised by the transducers below (the transducer on the left by the transducer on the right).



5. **Choosing the uniformisation.** Let $A = \{a, b, c\}$ be a totally ordered alphabet, where $a < b < c$, and let θ be the rational relation from A^* into itself whose graph is:

$$\widehat{\theta} = (a,a)^*\,(b,1)^*\,(1,b) \,\cup\, (a,1)^*\,(b,a)^*\,(1,c) \ .$$

Show that neither the radix uniformisation θ_{rad} nor the lexicographic selection θ_{lex} are rational functions.

6. **Inherently ambiguous rational relation.** Let $\mathcal{V}_1$ and $\mathcal{W}_1$ be the transducers of Example IV.2:

$$|\mathcal{V}_1| = \{(a^n b^m, c^n) \mid n, m \in \mathbb{N}\} \qquad \text{and} \qquad |\mathcal{W}_1| = \{(a^n b^m, c^m) \mid n, m \in \mathbb{N}\} \ .$$

Show that the rational relation $|\mathcal{V}_1| \cup |\mathcal{W}_1|$ is inherently ambiguous.